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The Electrodynamics and Statistical Mechanics of Linear Plasma Response Functions

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The purpose of this paper is to review and to extend, wherever possible, the Kramers–Kronig relations, sum rules, and symmetry properties for the electrodynamic transport tensors of a linear plasma medium. For complete generality, we consider both nonrelativistic and relativistic plasmas with and without external magnetic fields. Our study is carried out first within the framework of classical electrodynamics. We then exploit the statistical-mechanical fluctuation-dissipation theorem to further obtain the Onsager symmetry relations and Kubo sum-rule frequency moments. Of special significance is the emergence of a variety of new Kramers–Kronig formulae and *f*-sum rules for the inverse dispersion tensor.

KEY WORDS: plasmas; linear response functions; response functions; dielectric function; electrodynamics; Kramer–Kronig relations; sum rules; Onsager relation; fluctuation-dissipation theorem; transverse interaction.

1. INTRODUCTION

The conductivity, dielectric, and inverse dispersion tensors are the fundamental transport coefficients which portray the linear response of a plasma to a small external perturbing agency. Our first objective is to review and to extend, wherever possible, the Kramers–Kronig formulas, sum rules, and symmetry properties for these coefficients. Our second goal is to review the equilibrium fluctuation-dissipation theorem (FDT) applied to classical plasmas and to analyze the consequences of the relations ensuing from this theorem for the transport coefficients. The relations which we

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present are general, inasmuch as they are formulated for both nonrelativistic and relativistic plasmas with and without constant external magnetic fields. The motivation for this analysis lies in the central role played by the dielectric function in the description of plasma phenomena. Since the pioneering work of Nozières and Pines,[1a] Rukhadze and Silin,[1b] and Englert and Brout,[1c] it is now well understood that the frequency- and wave-number-dependent dielectric function incorporates a tremendous amount of information relevant in a wide range of physical effects (excitation of collective modes, energy loss of a test particle, scattering and absorption of electromagnetic waves, etc.), influences the structure of the plasma kinetic equation, and determines, via the pair correlation function, the equilibrium properties of the system. An understanding of the symmetry and analytic properties and of the interrelations among the transport coefficients is an indispensible asset in the algebraic manipulations, and these properties are of considerable interest themselves. The relations become rather intricate in the event the plasma is situated in an external magnetic field, where the dielectric and associated functions assume their full tensor character. It is in this connection, in particular, that we list results which have not been reported previously.

In many cases, the link between the quantity of primary physical interest and the dielectric function is provided by the fluctuation-dissipation theorem. This latter has, of course, a long history, both in its equilibrium version, first pronounced by Nyquist,[2a] Callen and Welton,[2b] and Kubo,[2c] and in its nonequilibrium variant, which was formulated by Rosenbluth and Rostoker,[3a] and Klimontovich and Silin.[3b] It is the nonequilibrium FDT which is of special importance in plasma physics, partly because of the great variety of nonequilibrium phenomena prevalent in plasmas, partly because the equilibrium theorem does not allow for a clear separation of electronic and ionic contributions even in equilibrium. Nevertheless, in this paper, we restrict ourselves to the discussion of the equilibrium theorem. We have several reasons for doing so. First, it is the equilibrium theorem that can be formulated in full generality, without evoking any perturbation-theoretic approach; moreover, the understanding of its range of validity and of its limitations yields the paradigm for the nonequilibrium theorem. Second, in equilibrium phenomena, it has, obviously, an interest of its own. Finally, the derivation of the nonequilibrium theorem would necessitate an entirely different approach, which would definitely be out of place in this paper.

Broadly speaking, this paper is divided into two parts. The first part surveys our objective mostly within the framework of classical electrodynamics.[1b,4–9] Here, we start with the "vacuum" and "medium" forms of Maxwell's equations together with the causul constitutive relations for a plasma. A comparison of these two forms[2]

[2] For a phenomenological description of plasmas, see Neufeld.[11] Neufeld writes the constitutive plasma equations in the form $\mathbf{P} = \alpha_{ee}\mathbf{E} + \alpha_{em}\mathbf{B}$, $\mathbf{M} = \alpha_{me}\mathbf{E} + \alpha_{mm}\mathbf{B}$, where $\mathbf{E}$ is the electric intensity, $\mathbf{B}$ the magnetic induction, $\mathbf{P}$ and $\mathbf{M}$ the electric and magnetic polarizations, and α_{ee}, α_{em}, α_{me}, and α_{mm} the appropriate transport coefficients. He expressly notes that the constitutive relations differ markedly from the following relations for molecular media: $\mathbf{P} = \chi_e\mathbf{E}$, $\mathbf{M} = \chi_m\mathbf{B}$, where χ_e and χ_m are the electric and magnetic susceptibilities. Our analysis shows that no such distinction should be made.

then results in the definition of the dielectric tensor.[1b,8,10–12] This tensor is Hermitian if the medium is nonabsorbing.[1b,8] The generalized Kramers–Kronig formulas and subsequent sum rules that these objects obey follow from the usual analyticity arguments posed for plus functions.[4,7]

In the second part, we use the powerful statistical-mechanical method of Kubo[2c,13–15] to develop the fluctuation-dissipation theorem for classical nonrelativistic and relativistic equilibrium plasmas. In formulating the Hamiltonian including interaction, the state of the unperturbed (equilibrium) system is understood to be the combined state of the collection of plasma particles and electromagnetic field (see, e.g., Refs. 16). For generality, the small external perturbing agency is cast in the form of a vector potential containing both the longitudinal and transverse components of the field.[3] It is shown that the relationship between the *external* conductivity (connecting the external electric-field perturbation to the current density of the plasma particles) and current correlation tensors is the same for nonrelativistic and relativistic plasmas. Finally, the FDT is used to formulate the well-known Onsager symmetry relations[17] and a set of sum-rule frequency moments for each of the relevant electrodynamic response functions. We shall then see that the lowest-order moment of each such set corresponds to a particular sum rule reported in the first part of this paper.

We remark that, for the derivation of the sum rules in Section 2, we have exploited the Onsager symmetry relations, which would imply an equilibrium hypothesis. However, the Onsager relations are, in fact, equally valid for homogeneous, stationary, nonequilibrium systems.

2. ELECTRODYNAMICS OF RESPONSE FUNCTIONS

Section 2 is divided into three subsections. In the first, the definitions of the dielectric, permeability, and conductivity tensors are established by comparing the "vacuum" and "medium" forms of Maxwell's equations. Then, a variety of response functions are catalogued and several among these are chosen for subsequent analysis in this paper; we also consider the physical significance of the transport coefficients in the process of power absorption by the plasma medium. The second and third subsections feature analyticity properties for the selected transport coefficients, their subsequent Kramers–Kronig relations, and sum rules.

The survey in this section is carried out mainly within the framework of classical electrodynamics, but with little reference to the evolution of the particle dynamics. Analysis and discussion of further relations depending on the actual particle dynamics is deferred to Section 3.

2.1. The Dielectric, Permeability, Electric and Magnetic Polarizability, and Conductivity Tensors

The electrodynamics of material media are conveniently described in terms of the total electric field intensity **E**, the magnetic induction **B**, the electric induction **D**,

[3] See also Ref. 15, pp. 446–449 for the linear response of a quantum-mechanical system to a vector potential as a perturbing agency.

and the magnetic field strength **H**. We adopt **E** and **B** as primary quantities. The relation between **E** and **B**, on the one hand, and **D** and **H** on the other, is, in general, fairly involved. In the present discussion, however, our basic assumption is that this relation is linear—which is certainly true for "weak" fields—permitting us to define proportionality factors which will be our fundamental entities.

We furthermore limit ourselves to an extended ("infinite") medium, homogeneous in space and stationary in time. The natural representation of physical variables in such a medium is in terms of Fourier transforms. The convention we adopt in this paper is the following: Let $f(\mathbf{r}, t)$ be any function of space and time and $f(\mathbf{k}, \omega)$ its Fourier transform. Then

$$f(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} \int_{L^3} d^3\mathbf{r}\, [\exp i(\omega t - \mathbf{k} \cdot \mathbf{r})] f(\mathbf{r}, t) \tag{1}$$

where L^3 is the large, but bounded volume of the system, and, in the inverse transform, the summation is made over the admissible set of discrete **k**-vectors.

Dielectric and Diamagnetic Tensors, Electric and Magnetic Polarizabilities. The electric induction **D** at a given space–time point $(\mathbf{r}, t)$ depends not only on the value of the total electric field intensity **E** at $(\mathbf{r}, t)$, but also on the value of **E** throughout the medium and at all previous times. In view of the linear approximation, the appropriate constitutive relation is written in the convolution form[1b]

$$\mathbf{D}(\mathbf{r}, t) = \int_{L^3} d^3\mathbf{r}' \int_{-\infty}^{t} dt'\, \boldsymbol{\varepsilon}(\mathbf{r} - \mathbf{r}', t - t', \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{r}', t') \tag{2}$$

where $\boldsymbol{\varepsilon}(\mathbf{r} - \mathbf{r}', t - t', \mathbf{B}_0)$ is the dielectric tensor of the medium.[4] Similarly, the magnetic field strength **H** and the magnetic induction **B** are assumed to be connected by the constitutive relation

$$\mathbf{H}(\mathbf{r}, t) = \int_{L^3} d^3\mathbf{r}' \int_{-\infty}^{t} dt'\, \boldsymbol{\nu}(\mathbf{r} - \mathbf{r}', t - t', \mathbf{B}_0) \cdot \mathbf{B}(\mathbf{r}', t') \tag{3}$$

where $\boldsymbol{\nu}(\mathbf{r} - \mathbf{r}', t - t', \mathbf{B}_0)$, is the inverse of the customary permeability tensor, and can be called the diamagnetic tensor of the medium. We have already exploited the fact that the medium is spatially homogeneous, by setting the kernels of (2) and (3) to depend on $(\mathbf{r}, \mathbf{r}')$ through the difference $(\mathbf{r} - \mathbf{r}')$. Equation (2) already implies the profound notion of causality, which states that the effect $\mathbf{D}(\mathbf{r}, t)$ cannot precede the cause $\mathbf{E}(\mathbf{r}', t')$. An alternative description is expressed by incorporating the causality condition into $\boldsymbol{\varepsilon}$ and $\boldsymbol{\nu}$:

$$\boldsymbol{\varepsilon}(\mathbf{r} - \mathbf{r}', t - t', \mathbf{B}_0) = 0 \tag{2a}$$

$$\boldsymbol{\nu}(\mathbf{r} - \mathbf{r}', t - t', \mathbf{B}_0) = 0 \qquad \text{for} \quad t < t' \tag{3a}$$

[4] If not otherwise stated, the plasma is specified to be in a constant external magnetic field. The existence of this field is referred to by inserting its symbol, $\mathbf{B}_0$, into the argument of the transport coefficient.

Then, clearly, the time integration in (2) and (3) can be extended to $t' = +\infty$ without changing the values of the integrals. Exploiting this fact, the Fourier-transformed linear constitutive relations for a homogeneous plasma will be

$$\mathbf{D}(\mathbf{k}, \omega) = \boldsymbol{\varepsilon}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega) \tag{4}$$

$$\mathbf{H}(\mathbf{k}, \omega) = \boldsymbol{\nu}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{B}(\mathbf{k}, \omega) \tag{5}$$

The electromagnetic behavior of the plasma can be described in two completely equivalent ways.[(8)] One first writes down the "vacuum" form of Maxwell's equations and expresses the total charge and current densities as $(\rho + \hat{\rho})$ and $(\mathbf{j} + \hat{\mathbf{j}})$, respectively, where $\hat{\rho}$ and $\hat{\mathbf{j}}$ are external sources and ρ and $\mathbf{j}$ are due to the plasma particles.[5] Next, one writes down Maxwell's equations for polarizable media by considering only $\hat{\rho}$ and $\hat{\mathbf{j}}$ as sources of the fields. The dielectric and diamagnetic tensors are then understood to implicitly contain the effects of the plasma particles. Upon comparing the "vacuum" and "medium" equations, one finds that

$$\mathbf{k} \times \{\boldsymbol{\xi}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot [\mathbf{k} \times \mathbf{E}(\mathbf{k}, \omega)]\} + (\omega^2/c^2)\boldsymbol{\alpha}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega) = i\mu_0\omega\mathbf{j}(\mathbf{k}, \omega) \tag{6}$$

$$\mathbf{k} \cdot [\boldsymbol{\alpha}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega)] = (i/\epsilon_0)\rho(\mathbf{k}, \omega) \tag{7}$$

where

$$\boldsymbol{\alpha}(\mathbf{k}, \omega, \mathbf{B}_0) = \boldsymbol{\varepsilon}(\mathbf{k}, \omega, \mathbf{B}_0) - \mathbf{1}, \qquad \boldsymbol{\xi}(\mathbf{k}, \omega, \mathbf{B}_0) = \boldsymbol{\nu}(\mathbf{k}, \omega, \mathbf{B}_0) - \mathbf{1}$$

are the electric and magnetic polarizabilities, respectively. Equation (6) serves to define the dielectric and permeability tensors,[6] but this definition is not unambigous. Projecting out transverse and longitudinal contributors to the polarizabilities

$$\boldsymbol{\alpha} \equiv \begin{pmatrix} \boldsymbol{\alpha}^{TT} & \boldsymbol{\alpha}^{TL} \\ \boldsymbol{\alpha}^{LT} & \alpha^{LL} \end{pmatrix} \tag{8}$$

etc., the lack of uniqueness appears in connection with the transverse polarizabilities. The reason for this is that, while the longitudinal polarizability is uniquely determined by the longitudinal current (a charge density), the transverse current carries the responsibility for both the transverse electric polarizability *and* the magnetic polarizability. To remove this ambiguity and in order to determine $\boldsymbol{\xi}^{TT}$ and $\boldsymbol{\alpha}^{TT}$, an additional condition is needed, and it is usually *postulated*[(8)] to be one of the three possibilities:

$$\boldsymbol{\alpha}^{TT} = 0, \qquad \boldsymbol{\xi}^{TT} \neq 0 \tag{9a}$$

$$\boldsymbol{\alpha}^{TT} = \alpha^{LL}\mathsf{T}, \qquad \boldsymbol{\xi}^{TT} \neq 0 \tag{9b}$$

$$\boldsymbol{\xi} = 0, \qquad \boldsymbol{\alpha}^{TT} \neq 0 \tag{9c}$$

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[5] In this paper, we adopt the MKS system of units.

[6] Remark that Eqs. (6) and (7) are not independent, because of the continuity equation for the charge density.

In (9a), it is clear that $\mathfrak{a}$ is essentially longitudinal ($\mathfrak{a}^{LL}$, $\mathfrak{a}^{LT}$, $\mathfrak{a}^{TL} \neq 0$) and the pure transverse effects are contained in ξ. Postulate (9b) is used most frequently in electrodynamics. Here, the difference between the longitudinal and transverse effects is incorporated into ξ. Moreover, the form of the relationship between **D** and **E** is essentially the sa me for both longitudinal and transverse components. In (9c), the magnetic polarizability is set equal to zero, so that the the transverse responsibilities are shifted to $\mathfrak{a}$. Here, $\mathfrak{a}$ contains the *full* transverse and longitudinal effects.

One obtains the following relation between ξ^{TT} as calculated from (9a) and $\mathfrak{a}^{TT}$ as calculated from (9c):

$$\mathbf{k} \times \xi^{TT} \times \mathbf{k} = (\omega^2/c^2)\mathfrak{a}^{TT} \tag{10}$$

Or, if ξ^{TT} is calculated from (9b), then (10) is modified to read[7]

$$\mathbf{k} \times \xi^{TT} \times \mathbf{k} = (\omega^2/c^2)(\mathfrak{a}^{TT} - \alpha^{LL}T) \tag{11}$$

While the concept of the permeability corresponds better to the actual physical situation, the use of a single quantity, the dielectric tensor, simplifies the calculations, so that the latter choice is usually preferred. Thus, setting $\xi = 0$, Eqs. (6) and (7) can be contracted into the form

$$\mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega) = (i/\omega\epsilon_0)\mathbf{j}(\mathbf{k}, \omega) \tag{12}$$

In this paper, Eq. (12) is taken to be the appropriate definition of the dielectric tensor,[8,10] and (10) that of the magnetic polarizability.

Conductivities. An equally important quantity is the conductivity σ. The induced current $\mathbf{j}(\mathbf{r}, t)$ is assumed to be connected to the total electric field **E** by the linear spatial-temporal convolution form of Ohm's law:

$$\mathbf{j}(\mathbf{k}, \omega) = \sigma(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega) \tag{13}$$

An "external" conductivity $\hat{\sigma}$ can also be defined[8]:

$$\mathbf{j}(\mathbf{k}, \omega) = \hat{\sigma}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \hat{\mathbf{E}}(\mathbf{k}, \omega) \tag{14}$$

where $\hat{\mathbf{E}}(\mathbf{k}, \omega)$ is the external electric field determined by external sources (i.e., the field in the absence of plasma particles). Note that both σ and $\hat{\sigma}$ are causal in the same way as ε. This will be discussed in some detail further on. The notion of the external conductivity is crucial to the development of the FDT, for it is $\hat{\sigma}$, and not σ, which is related to the response of the system to the external perturbing agency.

Comparison of (13) with (12) yields the following relation between σ and $\mathfrak{a}$:

$$\sigma(\mathbf{k}, \omega, \mathbf{B}_0) = -i\omega\epsilon_0\mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0) \tag{15}$$

[7] When there is no external magnetic field ($\mathbf{B}_0 = 0$), then our Eq. (1) reduced to Lindhard's Eq. (1.6) in Ref. 10; see also Ref. 1b, Eq. (2.22).

[8] See, for example, Ref. 13b, p. 282, Eq. (11.43).

Dispersion Tensor. A further, fundamentally important quantity is the dispersion tensor

$$\mathsf{D}(\mathbf{k}, \omega, \mathbf{B}_0) = n^2\mathsf{T} - \boldsymbol{\varepsilon}(\mathbf{k}, \omega, \mathbf{B}_0) \tag{16}$$

where $n = kc/\omega$ is the index of refraction of the medium. In free space, D becomes

$$\Delta = n^2\mathsf{T} - 1 \tag{17}$$

which is essentially the wave operator.

The quantities $\boldsymbol{\varepsilon}$, $\mathfrak{a}$ (or v and $\boldsymbol{\xi}$), σ, and $\hat{\sigma}$ are "response functions" in the sense that they determine the response ($\mathbf{D}$, $\mathbf{j}$) of the system to an external perturbation ($\mathbf{E}$, $\hat{\mathbf{E}}$). By considering the physical quantities which could play the role of a "driving" quantity and those which could be regarded as „responding," one can catalogue the appropriate response functions as in Table I. Note that we have also featured $\mathbf{E}$ as a driving quantity in order to include Ohm's law (13) and the linear constitutive relation

$$\check{\mathbf{E}}(\mathbf{k}, \omega) = \mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega) \tag{18}$$

From (13), (14), and the causal relation connecting $\mathbf{E}$ and $\hat{\mathbf{E}}$, one can show that

$$\hat{\sigma}(\mathbf{k}, \omega, \mathbf{B}_0) = \sigma(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \Delta \tag{19}$$

or, equivalently,

$$\sigma^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) - \hat{\sigma}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) = (i/\epsilon_0\omega)\Delta^{-1} \tag{20}$$

If, in the causal relation relation connecting $\mathbf{E}$ and $\mathbf{j}$, one sets $\hat{\mathbf{j}} = 0$, the resulting equation will possess nontrivial solutions only if (Ref. 18, pp. 10–12; Ref. 19, pp. 20, 29, 31):

$$|\mathsf{D}(\mathbf{k}, \omega_{\mathbf{k}})| = 0 \tag{21}$$

which is the, by now, widely familiar, plasma dispersion relation, determining the spectrum $\omega = \omega_{\mathbf{k}}$ of collective oscillations.

Table I. Electrodynamic Response Functions[a]

	$\hat{\mathbf{j}}\ (\hat{\rho})$	$\hat{\mathbf{E}}\ (\hat{E}^L)$	$\mathbf{E}$
$\mathbf{j}\ (\rho)$	$\mathfrak{a} \cdot \mathsf{D}^{-1}\ (-\alpha^{LL}/\epsilon^{LL})$	$\hat{\sigma}$	σ
$\check{\mathbf{E}} = \mathbf{E} - \hat{\mathbf{E}}\ (\check{E}^L)$	$(i/\epsilon_0\omega)\ \mathsf{D}^{-1} \cdot \mathfrak{a} \cdot \Delta^{-1}$	$\mathsf{D}^{-1} \cdot \mathfrak{a}\ (-\alpha^{LL}/\epsilon^{LL})$	$\mathfrak{a}$
$\hat{\mathbf{E}}$	$(i/\epsilon_0)\ \Delta^{-1}/\omega$	1	—
$\mathbf{E}\ (E^L)$	$(i/\epsilon_0)\ \mathsf{D}^{-1}/\omega\ (-i/\epsilon_0\omega\epsilon^{LL})$	$\mathsf{D}^{-1} \cdot \Delta\ (1/\epsilon^{LL})$	1

[a] Parenthetical expressions denote corresponding uncoupled longitudinal terms when $\mathbf{B}_0 = 0$.

If there is no external magnetic field, the dielectric tensor is diagonal and (21) splits into the two independent dispersion relations [Ref. 19, p. 21, Eqs. (31.70)]:

$$\epsilon^{LL}(\mathbf{k}, \omega_{\mathbf{k}}) = 0 \tag{22}$$

$$\epsilon^{TT}(n, n(\omega)) = n^2 \tag{23}$$

Dissipation and Hermiticity. In general, $\boldsymbol{\varepsilon}(\mathbf{k}, \omega, \mathbf{B}_0)$, $\boldsymbol{\nu}(\mathbf{k}, \omega, \mathbf{B}_0)$, $\boldsymbol{\sigma}(\mathbf{k}, \omega, \mathbf{B}_0)$, etc. are complex quantities. In order to demonstrate the physical significance of the real and imaginary parts, one is led to consider the process of power absorption by the plasma (Ref. 18, pp. 104–106, 126–131; Ref. 20). The spectral density of power absorption in an externally driven plasma is given by

$$\begin{aligned} R(\mathbf{k}, \omega) &= \mathrm{Re}\{\mathbf{E}^*(\mathbf{k}, \omega) \cdot \boldsymbol{\sigma}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega)\} \\ &= \mathbf{E}^*(\mathbf{k}, \omega) \cdot \boldsymbol{\sigma}^{\vee}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega), \\ \boldsymbol{\sigma}^{\vee} &= \tfrac{1}{2}(\sigma + \sigma^{\dagger}), \end{aligned} \tag{24}$$

since only the real part of the product contributes to the total energy. This, in turn, immediately implies that the Hermitian part of σ (or the anti-Hermitian part of $\boldsymbol{\varepsilon}$) originates from the dissipative character of the medium. In an ideal lossless medium, σ is anti-Hermitian (as $\boldsymbol{\varepsilon}$ is Hermitian). Similar conclusions can be shown to apply to $\hat{\boldsymbol{\sigma}}$. In a freely oscillating system (oscillating with the complex frequency $\omega_{\mathbf{k}} = \nu_{\mathbf{k}} + i\gamma_{\mathbf{k}}$, appropriate for the wave vector $\mathbf{k}$) a properly defined spectral energy density $\overline{W}(\mathbf{k})$ and the density of power absorption are expected to be linked by the relation

$$2\gamma_{\mathbf{k}}\overline{W}(\mathbf{k}) = -R(\mathbf{k}), \qquad R(\mathbf{k}) = R(\mathbf{k}, \omega_{\mathbf{k}}) \tag{25}$$

This implies that, for slight damping, the suitable expression for the spectral energy density is

$$\begin{aligned} \overline{W}(\mathbf{k}) &= W(\mathbf{k}) + \frac{i}{2}\,\mathbf{E}^*(\mathbf{k}) \cdot \left(\frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \omega}\right)_{\omega=\nu_{\mathbf{k}}} \cdot \mathbf{E}(\mathbf{k}) \\ &= \frac{\epsilon_0}{2}\,\mathbf{E}^*(\mathbf{k}) \cdot \frac{\partial}{\partial \omega}\,[\omega \boldsymbol{\varepsilon}^{\vee}(\mathbf{k}, \omega)] \cdot \mathbf{E}(\mathbf{k})\Big|_{\omega=\nu_{\mathbf{k}}} + \frac{1}{2\mu_0}\,\mathbf{B}^*(\mathbf{k}) \cdot \mathbf{B}(\mathbf{k}) \\ &= \frac{\epsilon_0}{2\omega}\,\mathbf{E}^*(\mathbf{k}) \cdot \frac{\partial}{\partial \omega}\,[\omega^2 \boldsymbol{\varepsilon}^{\vee}(\mathbf{k}, \omega)] \cdot \mathbf{E}(\mathbf{k})\Big|_{\omega=\nu_{\mathbf{k}}}, \end{aligned} \tag{26}$$

while

$$\begin{aligned} W(\mathbf{k}) &= W(\mathbf{k}, \nu_{\mathbf{k}}) \\ W(\mathbf{k}, \omega) &= \tfrac{1}{2}\epsilon_0 \mathbf{E}^*(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) + \tfrac{1}{2}\mu_0^{-1}\mathbf{B}^*(\mathbf{k}, \omega) \cdot \mathbf{B}(\mathbf{k}, \omega) \end{aligned} \tag{27}$$

is the vacuum expression for the spectral energy density.

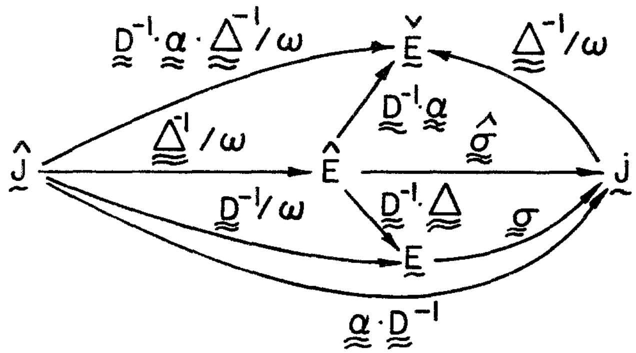

Fig. 1. Causal diagram.

2.2. Analyticity and the Kramers–Kronig Relations

The causality conditions [see, for example Eqs. (2a), (3a)] for the transport coefficients in the t-domain lead to important consequences with respect to their analytic behavior in the ω-plane. These and their subsequent Kramers–Kronig relations have been discussed at some length by Landau and Lifshitz,(4a,b) Pines,(7a,b) Martin,(9) and others. In laying the groundwork for our study, it seems to be most convenient to regard the ordinary conductivity as being the primary object, and we shall first derive its Kramers–Kronig relations. The analyticity properties of σ together with (15) then lead successively to the corresponding analyticity properties and Kramers–Kronig formulas for a, D^{-1}/ω (and, consequently, Δ^{-1}/ω), $\mathsf{a}\cdot\mathsf{D}^{-1}$, and $\hat{\sigma}$.

The causal diagram in Fig. 1 is presented to show how the different transport coefficients are interrelated. Note that an arrow points from a "cause" to an "effect" [see, for example, Eqs. (13) and (14)].

Conductivity. We start from Ohm's law for the ordinary conductivity in a spatially homogeneous plasma:

$$\mathbf{j}(\mathbf{k}, t) = \int_0^\infty d\tau\, \sigma(\mathbf{k}, \tau, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, t - \tau) \tag{28}$$

No current can flow at the moment of inception of the **E**-field[9] (at $t' = -\infty$ or at $\tau = t - t' = +\infty$), so that $\sigma(\mathbf{k}, \tau = \infty, \mathbf{B}_0) = 0$. Since there is no finite **E**-field capable of inducing infinite current, it follows that $\sigma(\mathbf{k}, \tau, \mathbf{B}_0)$ must be bounded. Hence, the integral

$$\int_0^\infty d\tau\, \sigma(\mathbf{k}, \tau, \mathbf{B}_0) \equiv \sigma(\mathbf{k}, \omega = 0, \mathbf{B}_0) \tag{29}$$

[9] The validity of this statement is not quite obvious, since **j** and **E** are not independent of each other. A careful analysis of the requirements involved has recently been given (see Ref. 9). It seems that the truth of this assumption should not be questioned for a classical plasma.

must also be bounded.[10] Then, regarding $\sigma(\mathbf{k}, \omega, \mathbf{B}_0)$ as a function of the complex variable $\omega = \omega' + i\omega''$, we can write

$$\sigma(\mathbf{k}, \omega' + i\omega'', \mathbf{B}_0) = \int_0^\infty d\tau\, (\exp i\omega'\tau)(\exp -\omega''\tau)\, \sigma(\mathbf{k}, \tau, \mathbf{B}_0) \tag{30}$$

Since the integral (29) is bounded and since the factor $e^{-\omega''\tau}$ enhances the convergence of the integral (29), it follows that (30) also converges. Hence, $\sigma(\mathbf{k}, \omega, \mathbf{B}_0)$ must be single-valued and regular in the upper-half ω-plane including $\omega = 0$.[4a,b] Moreover, $\sigma(\mathbf{k}, \omega, \mathbf{B}_0)$ has no singularities on the real axis. By definition, any function of the real variable ω whose analytic continuation on the upper-half ω-plane is analytic is a plus function. Therefore,

$$\sigma(\mathbf{k}, \omega, \mathbf{B}_0) \equiv \sigma_+(\mathbf{k}, \omega, \mathbf{B}_0) = \int_{-\infty}^{\infty} d\omega'\, \delta_+(\omega - \omega')\, \sigma(\mathbf{k}, \omega', \mathbf{B}_0) \tag{31}$$

Equation (31) can be restated in the form

$$\sigma(\mathbf{k}, \omega, \mathbf{B}_0) = \frac{i}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'}\, \sigma(\mathbf{k}, \omega', \mathbf{B}_0) \tag{32}$$

where $\mathscr{P}$ denotes the Cauchy principal part. Then, denoting the real and imaginary parts of σ by σ' and σ'', respectively, one readily obtains the Kramers–Kronig relations from (32)

$$\sigma'(\mathbf{k}, \omega, \mathbf{B}_0) = -\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'}\, \sigma''(\mathbf{k}, \omega', \mathbf{B}_0) \tag{33a}$$

$$\sigma''(\mathbf{k}, \omega, \mathbf{B}_0) = +\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'}\, \sigma'(\mathbf{k}, \omega', \mathbf{B}_0) \tag{33b}$$

Electric Polarizability. To obtain the corresponding formulas for the polarizability α, one can simply use (15) to replace σ in (33a, b) by α. One immediately obtains

$$\alpha''(\mathbf{k}, \omega, \mathbf{B}_0) = -\frac{1}{\pi\omega} \int_{-\infty}^{\infty} d\omega'\, \alpha'(\mathbf{k}, \omega', \mathbf{B}_0) + \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'}\, \alpha'(\mathbf{k}, \omega', \mathbf{B}_0) \tag{34a}$$

$$\alpha'(\mathbf{k}, \omega, \mathbf{B}_0) = -\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'}\, \alpha''(\mathbf{k}, \omega', \mathbf{B}_0) \tag{34b}$$

Then, from the Kramers–Kronig relation

$$\sigma'(\mathbf{k}, \omega = 0, \mathbf{B}_0) = \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega'}\, \sigma''(\mathbf{k}, \omega', \mathbf{B}_0) = -\frac{\epsilon_0}{\pi} \int_{-\infty}^{\infty} d\omega'\, \alpha'(\mathbf{k}, \omega', \mathbf{B}_0)$$

[10] In Appendix A, we discuss the $\omega \to 0$ behavior of σ, $\hat{\sigma}$, and D^{-1} for the following plasma models: (a) Warm, collisionless electron plasma.[21] (b) Cold, collisional ion–electron plasma with constant collision frequency and fixed scattering centers (see Ref. 18, p. 21). (c) Freely drifting, cold, collisional ion–electron plasma with constant collision frequency.[22]

we can write (34a) as[11]

$$\mathfrak{a}''(\mathbf{k}, \omega, \mathbf{B}_0) = \frac{1}{\epsilon_0 \omega} \sigma'(\mathbf{k}, \omega = 0, \mathbf{B}_0) + \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \mathfrak{a}'(\mathbf{k}, \omega', \mathbf{B}_0) \tag{34c}$$

Equations (34b, c) reveal that the modified polarizability

$$\mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0) - (i/\epsilon_0 \omega)\sigma(\mathbf{k}, \omega = 0, \mathbf{B}_0) \tag{35}$$

is a plus function. The analytic behavior of σ over the entire upper half-plane including the real axis has already been established; apparently, different considerations apply to the electric polarizability $\mathfrak{a}$. The reason for this is that the longitudinal dc conductivity of a plasma (which is a conducting medium) is expected to be finite. Correspondingly, the longitudinal polarizability should have a pole at $\omega = 0$: This will profoundly affect the analytic behavior of α^{LL}. However, whether an actual expression at hand for α^{LL} indeed behaves in this fashion depends on how sound the model espoused for the calculation is in the $\omega = 0$ frequency range; clearly, no collisionless-plasma model can claim to be correct in this region, and, thus, expressions, for α^{LL} calculated from the collisionless Vlasov equation will not be encumbered by this pathological behavior; in general, however, only (35) can be a candidate for Kramers–Kronig relations. The nonlongitudinal elements of $\mathfrak{a}$ might or might not have a singularity at $\omega = 0$ (see footnote 10); since the dc transverse field does not have a clear physical meaning, one cannot evoke the previously employed argument without further qualifications. In the $\mathbf{k} \to 0$ limit, however, σ^{TT} and σ^{LL} should be identical and exhibit the same singularity. A higher-order singularity should emerge in σ^{TT} in the case of magnetically polarizable media [$\xi(\omega = 0) \neq 0$]; this is a consequence of (10). Classical equilibrium plasmas are, however, not magnetically polarizable, and the subtraction procedure according to (35) will be satisfactory for any element of $\mathfrak{a}$.

We note that the surviving elements of $\sigma(\mathbf{k}, \omega = 0, \mathbf{B}_0)$ (referred to here as $\tilde{\sigma}_{\mu\nu}$) are all real for the following reason: Suppose that some of these elements are odd in ω as ω tends to zero. This implies that $\tilde{\sigma}_{\mu\nu}$ is either zero or infinite. But, by definition, $\tilde{\sigma}_{\mu\nu}$ cannot be zero, nor can it be infinite, since there is no finite electric field capable of inducing infinite current. Hence, each of the surviving elements must be even in ω as ω tends to zero. This admits that $\tilde{\sigma}_{\mu\nu} = \text{const}$ at $\omega = 0$. Moreover, only the real part of σ is even. Hence, the surviving elements of $\sigma(\mathbf{k}, \omega = 0, \mathbf{B}_0)$ are *real* constants (see footnote 10). The same conclusion obviously holds for the external conductivity.

To deduce the analyticity properties of D^{-1}/ω (which is the response function of the electric field for external current), $\mathfrak{a} \cdot \mathsf{D}^{-1}$ (which is the response function of the current to an external current), and $\mathfrak{a}$, we start by noting, from (15) [or from the "plus-ness" of (35)] that $\mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0)$ and, consequently, both $\mathsf{D}(\mathbf{k}, \omega, \mathbf{B}_0) = \Delta(\mathbf{k}, \omega) - \mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0)$ and adj $\mathsf{D}(\mathbf{k}, \omega, \mathbf{B}_0)$ (which is formed from products of elements of D) are analytic in the upper half-plane excluding the origin. Then, since $\mathsf{D}^{-1} =$

[11] The longitudinal components of the Kramers–Kronig formulas (34b,c) are also quoted for metals in Ref. 4a, p. 260, Eqs. (62.8, 11).

adj $\mathsf{D}/|\mathsf{D}|$, it is clear that the only poles of $\mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)$ are those which correspond to collective modes ($|\mathsf{D}| = 0$), and, for a stable system, such poles must necessarily lie in the lower-half ω-plane. Therefore, $\mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)$, and, consequently, the objects $\mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)/\omega$, $\mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)$, and $\hat{\mathfrak{o}}$ [see Eq. (19)] are regular and one-valued in the upper half-plane. The behavior of these objects at $\omega = 0$ will be discussed as we consider each of them separately.

The Electric Field–External Current Response Function. Starting with D^{-1}/ω, we note that, as ω tends to zero, the longitudinal component,

$$\mathbf{k} \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{k}/\omega,$$

can diverge as $1/\omega$ (see footnote 10) due to the infinite longitudinal electric field (piling up of charge) associated with a finite external current density. The singularity of this transport coefficient at $\omega = 0$ can be eliminated by subtracting off

$$\mathsf{D}^{-1}(\mathbf{k}, 0, \mathbf{B}_0)/\omega.$$

The resulting object

$$[\mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) - \mathsf{D}^{-1}(\mathbf{k}, 0, \mathbf{B}_0)]/\omega \tag{36}$$

is thus analytic in the upper half-plane including the origin. Moreover, as ω tends to infinity, (36) tends to zero at least as rapidly as $1/\omega$. Clearly, (36) is a plus function, and therefore satisfies the Kramers–Kronig relations

$$\operatorname{Re} \frac{\mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) - \mathsf{D}^{-1}(\mathbf{k}, 0, \mathbf{B}_0)}{\omega} = -\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \operatorname{Im} \frac{\mathsf{D}^{-1}(\mathbf{k}, \omega', \mathbf{B}_0) - \mathsf{D}^{-1}(\mathbf{k}, 0, \mathbf{B}_0)}{\omega'} \tag{37a}$$

$$\operatorname{Im} \frac{\mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) - \mathsf{D}^{-1}(\mathbf{k}, 0, \mathbf{B}_0)}{\omega} = +\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \operatorname{Re} \frac{\mathsf{D}^{-1}(\mathbf{k}, \omega', \mathbf{B}_0) - \mathsf{D}^{-1}(\mathbf{k}, 0, \mathbf{B}_0)}{\omega'} \tag{37b}$$

In particular, when there is no external magnetic field acting, Eqs. (37a, b) split into separate longitudinal and transverse relations:

Longitudinal

$$\operatorname{Re} \frac{1}{\omega} \left[\frac{1}{\epsilon^{LL}(\mathbf{k}, \omega)} - \frac{1}{\epsilon^{LL}(\mathbf{k}, 0)}\right] = -\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \operatorname{Im} \frac{1}{\omega' \epsilon^{LL}(\mathbf{k}, \omega')} \tag{38a}$$

$$\operatorname{Im} \frac{1}{\omega \epsilon^{LL}(\mathbf{k}, \omega)} = \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \operatorname{Re} \frac{1}{\omega'} \left[\frac{1}{\epsilon^{LL}(\mathbf{k}, \omega')} - \frac{1}{\epsilon^{LL}(\mathbf{k}, 0)}\right] \tag{38b}$$

Transverse

$$\operatorname{Re}\frac{1}{\omega D^{TT}(\mathbf{k},\omega)} = -\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega-\omega'}\operatorname{Im}\frac{1}{\omega' D^{TT}(\mathbf{k},\omega')} \tag{39a}$$

$$\operatorname{Im}\frac{1}{\omega D^{TT}(\mathbf{k},\omega)} = +\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega-\omega'}\operatorname{Re}\frac{1}{\omega' D^{TT}(\mathbf{k},\omega')} \tag{39b}$$

Should the plasma be absent, then (36) goes over into the retarded vacuum propagator

$$\frac{\Delta^{-1}(\mathbf{k},\omega) - \Delta^{-1}(\mathbf{k},0)}{\omega} = \frac{\omega}{k^2c^2-\omega^2}\mathsf{T} \tag{40}$$

with its familiar causality relations.

Note that, while the transverse component of the object $\Delta^{-1}(\mathbf{k},\omega)/\omega$ vanishes at $\omega = 0$, the longitudinal component, $-\mathbf{kk}/\omega$, nevertheless exhibits a divergence there due to the infinite *external* charge density associated with a finite external current density. This is what motivates the formation of the object (40). Moreover, it is this infinite external charge density which is actually responsible for the divergence in $\mathbf{k}\cdot\mathsf{D}^{-1}(\mathbf{k},\omega,\mathbf{B}_0)\cdot\mathbf{k}/\omega$ at $\omega = 0$.

External Conductivity. Concerning the external conductivity, we recall that the surviving elements of $\hat{\sigma}(\mathbf{k},\omega = 0,\mathbf{B}_0)$ are (real) constants.[12] Moreover, $\hat{\sigma}$ exhibits the same high-frequency behavior as σ [cf. Eq. (20)], so that, as ω tends to infinity, $\hat{\sigma}$ tends to zero at least as rapidly as $1/\omega$. These, together with the established analyticity of $\hat{\sigma}$ in the upper-half ω-plane, qualify $\hat{\sigma}$ to be a plus function. Consequently, it obeys the same Kramers–Kronig relations (33a, b) as σ.

The Current–External Current Response Function. Turning next to the object $\mathfrak{a}\cdot\mathsf{D}^{-1}$, which connects the "driving" quantity $\hat{\mathbf{j}}$ to the "responding" quantity $\mathbf{j}$ (cf. Table I), $\mathfrak{a}\cdot\mathsf{D}^{-1}$ must tend to a constant as ω tends to zero (see footnote 12), since a finite $\hat{\mathbf{j}}$ cannot produce an infinite $\mathbf{j}$ (cf. Appendix A). This, and the fact that this object is analytic in the upper half-plane and tends to zero as ω tends to infinity at least as rapidly as $1/\omega^2$, are sufficient to qualify $\mathfrak{a}\cdot\mathsf{D}^{-1}$ as a plus function.

[12] One can be more precise about the $\omega\to 0$ behavior of $\hat{\sigma}$. From Table I, it can be shown that

$$\mathbf{j}^L(\mathbf{k},\omega\to 0)\to -(i/\epsilon_0\omega)\,\hat{\sigma}^{LL}(\mathbf{k},\omega\to 0,\mathbf{B}_0)\cdot\hat{\mathbf{j}}^L(\mathbf{k},\omega\to 0)$$

$$\mathbf{j}^T(\mathbf{k},\omega\to 0)\to -(i/\epsilon_0\omega)\,\hat{\sigma}^{TL}(\mathbf{k},\omega\to 0,\mathbf{B}_0)\cdot\hat{\mathbf{j}}^L(\mathbf{k},\omega\to 0)$$

Since a finite $\hat{\mathbf{j}}^L$ cannot induce infinite $\mathbf{j}$, it follows that, at most,

$$\hat{\sigma}^{LL}(\mathbf{k},\omega\to 0,\mathbf{B}_0)\to ia\epsilon_0\omega,\qquad \hat{\sigma}^{TL}(\mathbf{k},\omega\to 0,\mathbf{B}_0)\to i\mathsf{b}\epsilon_0\omega$$

where a and b are real scalar and tensor constants. [Consequently, one can also deduce that, as $\omega\to 0$,

$$\mathfrak{a}\cdot\mathsf{D}^{-1}\to -(i/\epsilon_0\omega)(\hat{\sigma}^{TL}+\hat{\sigma}^{LL}\,\mathsf{1})\cdot\mathbf{kk} = (\mathsf{b}+a\mathsf{1})\cdot\mathbf{kk}]$$

Its Kramers–Kronig relations are obviously

$$\mathrm{Re}[\mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)] = -\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \mathrm{Im}[\mathfrak{a}(\mathbf{k}, \omega', \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega', \mathbf{B}_0)] \tag{41a}$$

$$\mathrm{Im}[\mathfrak{a}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)] = +\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \mathrm{Re}[\mathfrak{a}(\mathbf{k}, \omega', \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega', \mathbf{B}_0)] \tag{41b}$$

These formulas can also be stated for the longitudinal and transverse projection separately. Some manipulation yields the longitudinal formulas

$$\begin{aligned} 1 + \mathbf{k} \cdot \mathsf{D}^{-1\vee}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{k} &= \frac{i}{\pi} \mathscr{P} \int_{\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \mathbf{k} \cdot \mathsf{D}^{-1\wedge}(\mathbf{k}, \omega', \mathbf{B}_0) \cdot \mathbf{k} \\ \mathbf{k} \cdot \mathsf{D}^{-1\wedge}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{k} &= \frac{i}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} [1 + \mathbf{k} \cdot \mathsf{D}^{-1\vee}(\mathbf{k}, \omega', \mathbf{B}_0) \cdot \mathbf{k}] \end{aligned} \tag{42}$$

In particular, when there is no external magnetic field, Eqs. (41a,b) split into the independent longitudinal and transverse relations:

Longitudinal[7a,b]

$$\mathrm{Re} \frac{\alpha^{LL}(\mathbf{k}, \omega)}{\epsilon^{LL}(\mathbf{k}, \omega)} = -\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \mathrm{Im} \frac{\alpha^{LL}(\mathbf{k}, \omega')}{\epsilon^{LL}(\mathbf{k}, \omega')} \tag{43a}$$

$$\mathrm{Im} \frac{\alpha^{LL}(\mathbf{k}, \omega)}{\epsilon^{LL}(\mathbf{k}, \omega)} = +\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \mathrm{Re} \frac{\alpha^{LL}(\mathbf{k}, \omega')}{\epsilon^{LL}(\mathbf{k}, \omega')} \tag{43b}$$

Transverse

$$\mathrm{Re} \frac{\alpha^{TT}(\mathbf{k}, \omega)}{D^{TT}(\mathbf{k}, \omega)} = -\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \mathrm{Im} \frac{\alpha^{TT}(\mathbf{k}, \omega)}{D^{TT}(\mathbf{k}, \omega')} \tag{44a}$$

$$\mathrm{Im} \frac{\alpha^{TT}(\mathbf{k}, \omega)}{D^{TT}(\mathbf{k}, \omega)} = +\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \mathrm{Re} \frac{\alpha^{TT}(\mathbf{k}, \omega')}{D^{TT}(\mathbf{k}, \omega')} \tag{44b}$$

Magnetic Polarizability. A response function not related to the reaction of the system to an external field, and therefore not represented in our Table I or Fig. 1, is the magnetic polarizability tensor ξ^{TT}. From (10), we see that ξ^{TT} is analytic in the upper-half ω-plane and vanishes at $\omega = 0$ [cf. Eq. (29) and our discussion below (35) in connection the behavior of σ^{TT} as $\omega \to 0$]. However, as ω tends to infinity,

$$\{\mathbf{k} \times \xi^{TT} \times \mathbf{k}\}_{11,22} \to \frac{{\omega_0}^2}{k^2c^2}, \qquad \{\mathbf{k} \times \xi^{TT} \times \mathbf{k}\}_{12} \to 0 \qquad \text{as} \quad 1/\omega$$

so that the diagonal element of $\xi^{TT}(\mathbf{k}, \omega, \mathbf{B}_0)$ alone cannot be plus functions. On the other hand, all elements of the modified magnetic polarizability tensor

$$\xi^{TT}(\mathbf{k}, \omega, \mathbf{B}_0) - \xi^{TT}(\mathbf{k}, \infty, \mathbf{B}_0) \tag{45}$$

do tend to zero as ω tends to infinity. Clearly, the object (45) is a plus function, and therefore satisfies a set of Kramers–Kronig relations.

2.3. Sum Rules

The sum rules which we now derive give definite values to integrals of frequency-weighted dissipative parts of the transport tensors. For the most part, these rules are the same for classical and quantum plasmas, and they are formulated quite independently of the particle interactions. In this section, we shall discuss sum rules for $\mathfrak{a}$, σ, $\hat{\sigma}$, $1/\epsilon^{LL}$, and α^{TT}/D^{TT}; some of these are of general knowledge, while others are more or less straightforward generalizations and extensions of the known cases. This study will be carried out starting from the Kramers–Kronig relations formulated in the preceding section. Such an approach exploits the fact that, at high frequencies, the electrons behave like a collection of noninteracting particles. In a nonrelativistic plasma, this implies that,[13] for $\omega \to \infty$, $\mathfrak{a}$ is diagonal and $\alpha'_{11} \approx \alpha'_{22} \approx \alpha'_{33} \approx -\omega_0{}^2/\omega^2$, or

$$\sigma''_{11} \approx \sigma''_{22} \approx \sigma''_{33} \approx \epsilon_0 \omega_0{}^2/\omega \tag{46}$$

where $\omega_0 = (n_0 e^2/m\epsilon_0)^{1/2}$ is the plasma frequency and n_0 the equilibrium number density.

In a nonabsorbing medium, the dielectric tensor is Hermitian, so that its symmetric components $\epsilon_{(\mu\nu)}$ must be real. Then, clearly, the presence of any dissipation is reflected by an imaginary contribution to each such component. On the other hand, the antisymmetric elements $\epsilon_{[\mu\nu]}$ are pure imaginary in a nonabsorbing medium, so that, for these latter elements, it is now the real part which reflects dissipation. Without loss of generality, we choose either the $\mathbf{B}_0$-system $[\mathbf{k}{=}\mathbf{k}_x, 0, k_z), \mathbf{B}_0 = (0, 0, \mathbf{B}_0)]$ or the $\mathbf{k}$-system $[\mathbf{k} = (0, 0, k), \mathbf{B}_0 = (\mathbf{B}_{0x}, 0, \mathbf{B}_{0z})]$ for the relative orientations of the external field and wave vector. Equation (176) in Section 3 then shows that ϵ_{13} and ϵ_{31} are symmetric, while the remaining off-diagonal components are antisymmetric.

Conductivity Sum Rules. Consider first the elements 11, 22, 33, 13, and 31 of the polarizability tensor. In the high-frequency limit ($\omega \to \infty$), a denominator expansion of the Kramers–Kronig relations (34b) leads to

$$\alpha'_{(\mu\nu)}(\mathbf{k}, \omega \to \infty, \mathbf{B}_0)$$
$$= -\frac{2}{\pi\omega^2}\int_0^\infty d\omega'\,\omega'\alpha''_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) - \frac{2}{\pi\omega^4}\int_0^\infty d\omega'\,\omega'^3\alpha''_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) - \cdots \tag{47}$$

[Ref. 4a, p. 261, Eq. (62.14); Ref. 7a, p. 136, Eq. (3.40)][14]. From the reality of

[13] See, for example, Ref. 4a, p. 251; Ref. 3, p. 104, Eq. (98); Ref. 9, p. 145; Ref. 23, p. 58.

[14] There are definite restrictions on the validity of the denominator expansion leading to Eq. (47), and these restrictions strongly depend on the particular plasma model chosen. To see this, suppose that, as ω tends to infinity, $\alpha''_{(\mu\nu)} \to A/\omega^{2s+1}$, where A is some constant and s can take on integral values from one on up (for example, in the constant-collision-frequency model of a cold plasma,[18] $s = 1$). We can then write

$$\lim_{\omega\to\infty} \mathscr{P}\int_{-\infty}^\infty \frac{d\omega'}{\omega - \omega'}\alpha''_{(\mu\nu)}(\omega') \equiv 2\lim_{\omega\to\infty}\left\{\mathscr{P}\int_0^\omega \frac{d\omega'}{\omega^2 - \omega'^2}\omega'\alpha''_{(\mu\nu)}(\omega') + \mathscr{P}\int_\omega^\infty \frac{d\omega'}{\omega^2 - \omega'^2}\omega'\alpha''_{(\mu\nu)}(\omega')\right\}$$

$\mathfrak{a}(\mathbf{r}, t, \mathbf{B}_0)$ and its invariance under spatial inversion,[15] α'_{13} must be an even function of ω. Thus, $\alpha'_{13}(\mathbf{k}, \omega \to \infty, \mathbf{B}_0)$ can be, at most, of order $1/\omega^4$ of smallness, whence,

$$\approx 2 \lim_{\omega\to\infty} \left(\sum_{n=0}^{s-1} \frac{1}{\omega^{2(n+1)}} \int_0^{\omega} d\omega' \, \omega'^{2n+1} \alpha''_{(\mu\nu)}(\omega') \right.$$

$$\left. + \sum_{n=s}^{\infty} \frac{1}{\omega^{2(n+1)}} \int_0^{\omega} d\omega' \, \omega'^{2n+1} \alpha''_{(\mu\nu)}(\omega') + A \mathscr{P} \int_{\omega}^{\infty} \frac{d\omega'}{\omega^2 - \omega'^2} \frac{1}{\omega'^{2s}} \right)$$

Since

$$2A \lim_{\omega\to\infty} \mathscr{P} \int_{\omega}^{\infty} \frac{d\omega'}{\omega^2 - \omega'^2} \frac{1}{\omega'^{2s}} \approx \lim_{\omega\to\infty} \frac{S_{(\mu\nu)}}{\omega^{2s+1}}$$

where $S_{(\mu\nu)}$ is a constant, it follows that this last term might be comparable with or larger than the terms in the second summation extending from s to infinity. More precisely, the integrals in the first summation ($\Sigma_{n=0}^{s-1}$) are bounded, whereas those in the second diverge in such a way that

$$2 \lim_{\omega\to\infty} \sum_{n=s}^{\infty} \frac{1}{\omega^{2(n+1)}} \int_0^{\omega} d\omega' \, \omega'^{2n+1} \alpha''_{(\mu\nu)}(\omega') \approx \lim_{\omega\to\infty} \frac{R_{(\mu\nu)}}{\omega^{2s+1}}$$

where $R_{(\mu\nu)}$ is a constant. Then the expansion can be written as

$$\lim_{\omega\to\infty} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \alpha''_{(\mu\nu)}(\omega') \approx \lim_{\omega\to\infty} \left\{ \sum_{n=0}^{s-1} \frac{I_{(\mu\nu)}^{(n)}}{\omega^{2(n+1)}} + \frac{R_{(\mu\nu)} + S_{(\mu\nu)}}{\omega^{2s+1}} \right\}$$

where $I_{(\mu\nu)}^{(n)} = 2 \int_0^{\omega} d\omega' \, \omega'^{2n+1} \alpha''_{(\mu\nu)}(\omega')$. For example, in the case of the constant-collision-frequency model, where $s = 1$, we have

$$\lim_{\omega\to\infty} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \alpha''_{(\mu\nu)}(\omega') \approx \frac{I_{(\mu\nu)}^{(0)}}{\omega^2} + \frac{R_{(\mu\nu)} + S_{(\mu\nu)}}{\omega^3}$$

which implies that, for $\alpha''_{(\mu\nu)}(\omega \to \infty) \to 1/\omega^3$, one cannot go beyond the first term on the r.h.s. of (47) in formulating the sum rules.

Turning next to the antisymmetric elements of $\mathfrak{a}$ ($\alpha'_{[\mu\nu]}(\omega \to \infty) \to 1/\omega^{2(s+1)}$, $s \geqslant 1$), it can be similarly shown that

$$\lim_{\omega\to\infty} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \alpha'_{[\mu\nu]}(\omega') \approx \lim_{\omega\to\infty} \left\{ \sum_{n=0}^{s} \frac{I_{[\mu\nu]}^{(n)}}{\omega^{2n+1}} + \frac{R_{[\mu\nu]} + S_{[\mu\nu]}}{\omega^{2(s+1)}} \right\}$$

where $I_{[\mu\nu]}^{(n)} = 2 \int_0^{\omega} d\omega' \, \omega'^{2n} \alpha'_{[\mu\nu]}(\omega')$. Then, for the particular case of the constant-collision-frequency model ($s = 1$), we have

$$\lim_{\omega\to\infty} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} \alpha'_{[\mu\nu]}(\omega') \approx \frac{I_{[\mu\nu]}^{(0)}}{\omega} + \frac{I_{[\mu\nu]}^{(1)}}{\omega^3} + \frac{R_{[\mu\nu]} + S_{[\mu\nu]}}{\omega^4},$$

which implies that, for the formulation of the relevant sum rules, one cannot go beyond the third term on the right-hand side of (50).

In a collisionless model (cf. Ref. 21) the dissipative part of $\mathfrak{a}$ vanishes exponentially following the high-velocity behavior of the distribution function, and therefore the high-frequency expansion is not restricted in any way. Higher-order approximations, however, being tantamount to the inclusion of collisions, have not been worked out in all detail, and are expected to lead to essentially the same high-frequency behavior[32] as the naive constant-collision-frequency model.

[15] For equilibrium systems, there is no relative drift velocity among the plasma particles.

comparison of coefficients of $1/\omega^2$ and of $1/\omega^4$ between (46) and (47) leads to

$$\int_0^\infty d\omega' \, \omega' \alpha''_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) = \tfrac{1}{2}\pi\omega_0{}^2 \, \delta_{\mu\nu} \tag{48}$$

From (15) and (48), one readily obtain the "conductivity" sum rule [see, e.g., Ref. 7b, p. 209, Eq. (4.25)]

$$\int_0^\infty d\omega' \, \sigma'_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) = \tfrac{1}{2}\pi\omega_0{}^2\epsilon_0\delta_{\mu\nu} \tag{49}$$

Turning next to the remaining off-diagonal elements $\mu\nu = 12, 21, 23, 32$, the high-frequency denominator expansion of the Kramers–Kronig relation (1.34) yields (see footnote 14)

$$\begin{aligned} \alpha''_{[\mu\nu]}(\mathbf{k}, \omega \to \infty, \mathbf{B}_0) = {} & (1/\epsilon_0\omega) \, \sigma_{[\mu\nu]}(\mathbf{k}, \omega = 0, \mathbf{B}_0) + (2/\pi\omega) \int_0^\infty d\omega' \, \alpha'_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) \\ & + (2/\pi\omega^3) \int_0^\infty d\omega' \, \omega'^2 \alpha'_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) + \cdots \end{aligned} \tag{50}$$

From the reality condition and invariance under spatial inversion, the component $\alpha''_{[\mu\nu]}$ is an odd function of the frequency, and therefore can be, at most, of order $1/\omega^3$ of smallness. Hence, from (50) one can deduce the sum rule

$$\int_0^\infty d\omega' \, \alpha'_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) = -(\pi/2\epsilon_0) \, \sigma_{[\mu\nu]}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \tag{51}$$

or, in terms of the conductivity,

$$\int_0^\infty d\omega' \, \{\sigma''_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0)/\omega'\} = \tfrac{1}{2}\pi\sigma_{[\mu\nu]}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \tag{52}$$

Equation (52) can also be obtained by setting $\omega = 0$ in the Kramers–Kronig relation (33a).

Further results can be obtained only by examining the high-frequency behavior of $\alpha''_{[\mu\nu]}$. To do this, one consider the collection of noninteracting motionless particles to be governed by the equation of motion

$$(\omega\delta_{\mu\alpha} - i\omega_c\epsilon_{\mu\nu\alpha}\hat{B}_{0\nu}) \, j_\alpha = i\epsilon_0\omega_0{}^2 E_\mu \qquad (\text{for} \quad \omega \to \infty), \tag{53}$$

where $\omega_c = |e| B_0/m$ is the cyclotron frequency, $\epsilon_{\mu\nu\alpha}$ is a component of the unit permutation pseudotensor, and $\hat{\mathbf{B}}_0$ is the unit vector in the direction of $\mathbf{B}_0$. Equations (12) and (53) then give

$$\alpha''_{[\mu\nu]}(\mathbf{k}, \omega \to \infty, \mathbf{B}_0) = (\omega_0{}^2\omega_c/\omega^3) \, \epsilon_{\mu\nu\alpha}\hat{B}_{0\alpha} \tag{54}$$

which, in virture of (50), yields the higher-order sum rules

$$\int_0^\infty d\omega' \, \omega'^2 \alpha'_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) = \tfrac{1}{2}\pi\omega_0{}^2\omega_c \, \epsilon_{\mu\nu\alpha}\hat{B}_{0\alpha} \tag{55}$$

with the corresponding conductivity rule

$$\int_0^\infty d\omega'\, \omega' \sigma''_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) = -\tfrac{1}{2}\pi\omega_0{}^2\omega_c\, \epsilon_0\epsilon_{\mu\nu\alpha}\hat{B}_{0\alpha} \tag{56}$$

Compressibility Sum Rules. Upon setting $\omega = 0$ in the Kramers–Kronig relation (34), one obtains

$$\mathscr{P}\int_0^\infty \frac{d\omega'}{\omega'}\, \alpha''_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) = \frac{\pi}{2}\, \alpha'_{(\mu\nu)}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \tag{57}$$

for the symmetric elements of the polarizability. Since all the *surviving* elements of $\sigma(\mathbf{k}, \omega = 0, \mathbf{B}_0)$ are real, then, as ω tends to zero, the corresponding $\sigma''_{\mu\nu}(\mathbf{k}, \omega = 0, \mathbf{B}_0)$ tend to zero at least as rapidly as ω (because of the odd parity in ω). This implies that the corresponding $\alpha'_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0)$ can, at most, tend to constant values as $\omega \to 0$, consistent with the even parity of $\alpha'_{\mu\nu}(\omega)$. In particular, for the case of a warm, collisionless plasma [see Appendices A and B—Eq. (B9)],

$$\int_0^\infty \frac{d\omega'}{\omega'}\, \mathrm{Im}\, \alpha^{LL}(\mathbf{k}, \omega', \mathbf{B}_0) = \frac{\pi}{2}\frac{\kappa^2}{k^2} \tag{58}$$

in the $\mathbf{k}$-system. Equation (58) is akin to the so-called "compressibility sum rule" discussed by Pines and Nozières for an electron liquid [Ref. 7b, p. 210, Eq. (4.29)].

Table II summarizes our sum-rule findings for the polarizability tensor. No sum rules exist for $n \geqslant 3$ due to the appearance of terms of the type $\omega_0{}^2k^2v^2/\omega^4$ in the expansion similar to (47). It is interesting to note that, for a warm, collisionless plasma in a constant external magnetic field, the Vlasov expression for $\mathbf{\alpha}$[19,21] completely exhausts the sum rules in Table II (see Appendix B).

Concerning the external conductivity, we observe that $\hat{\sigma}$ and σ both satisfy the same Kramers–Kronig relations and have the same asymptotic behavior at high [frequencies [cf. Eq. (1.20)]. Therefore, $\hat{\sigma}$ and σ obey the same sum rules, and, in order to obtain rules for $\hat{\sigma}$, one need merely replace σ everywhere in (49), (52), and (56) by $\hat{\sigma}$. Table III is presented as a summary for $\hat{\sigma}$. The rule corresponding to $n = -2$ was obtained by setting $\omega = 0$ in the Kramers–Kronig relation

$$\frac{\hat{\sigma}''(\mathbf{k}, \omega, \mathbf{B}_0)}{\omega} = \frac{1}{\pi}\mathscr{P}\int_{-\infty}^\infty \frac{d\omega'}{\omega^2 - \omega'^2}\, \hat{\sigma}'(\mathbf{k}, \omega', \mathbf{B}_0)$$

Table II. Polarizability Sum Rules

n	$(2/\pi)\int_0^\infty d\omega'\, \omega'^n\alpha''_{(\mu\nu)}(\omega')$	$(2/\pi)\int_0^\infty d\omega'\, \omega'^n\alpha'_{[\mu\nu]}(\omega')$
-1	$\alpha'_{(\mu\nu)}(\mathbf{k}, \omega = 0, \mathbf{B}_0)$	—
0	—	$-\epsilon_0^{-1}\sigma_{[\mu\nu]}(\mathbf{k}, \omega = 0, \mathbf{B}_0)$
1	$\omega_0{}^2\delta_{(\mu\nu)}$	—
2	—	$\omega_0{}^2\omega_c\epsilon_{[\mu\nu]\alpha}B_{0\alpha}$

Table III. External Conductivity Sum Rules

n	$(2/\pi)\int_0^\infty d\omega'\,\omega'^n\hat{\sigma}'_{(\mu\nu)}(\omega')$	$(2/\pi)\int_0^\infty d\omega'\,\omega'^n\hat{\sigma}''_{[\mu\nu]}(\omega')$
-2	$-\lim_{\omega\to 0}[\hat{\sigma}''_{(\mu\nu)}(\mathbf{k},\omega,\mathbf{B}_0/\omega]$	—
-1	—	$\hat{\sigma}_{[\mu\nu]}(\mathbf{k},\omega=0,\mathbf{B}_0)$
0	$\omega_0{}^2\epsilon_0\delta_{(\mu\nu)}$	—
1	—	$-\omega_0{}^2\omega_c\epsilon_0\epsilon_{[\mu\nu]\alpha}\hat{B}_{0\alpha}$

The boundedness of the dc $\hat{\sigma}$, together with the odd frequency parity of its imaginary part, ensure that $\lim_{\omega\to 0}\hat{\sigma}''_{(\mu\nu)}(\omega)/\omega$ is constant.

***f*-Sum Rules.** Another family of sum rules (the so-called *f*-sum rules[16]) are obtained by considering the quantities

$$\Delta\cdot\mathsf{D}^{-1}\cdot\Delta,\qquad \frac{\alpha^{LL}}{\epsilon^{LL}}\equiv 1-\frac{1}{\epsilon^{LL}},\qquad \frac{\alpha^{TT}}{D^{TT}}\equiv\frac{n^2-1}{D^{TT}}-1$$

First, from (15) and (19),

$$\hat{\sigma}(\mathbf{k},\omega,\mathbf{B}_0)=-i\omega\epsilon_0\Delta(\mathbf{k},\omega)\cdot\mathsf{D}^{-1}(\mathbf{k},\omega,\mathbf{B}_0)\cdot\Delta(\mathbf{k},\omega)+i\omega\epsilon_0\Delta(\mathbf{k},\omega)\tag{59}$$

so that the sum rules for $\Delta\cdot\mathsf{D}^{-1}\cdot\Delta$ are easily obtained from the sum rules for $\hat{\sigma}$. Upon exploiting the symmetry relations in Section 3 [cf. Eq. (176)], one finds, from (59), that

$$\hat{\sigma}'_{(\mu\nu)}(\mathbf{k},\omega,\mathbf{B}_0)=\omega\epsilon_0\Delta_{\mu\alpha}(\mathbf{k},\omega)\,\mathrm{Im}\,D^{-1}_{(\alpha\beta)}(\mathbf{k},\omega,\mathbf{B}_0)\,\Delta_{\beta\nu}(\mathbf{k},\omega)\tag{60}$$

$$\hat{\sigma}''_{[\mu\nu]}(\mathbf{k},\omega,\mathbf{B}_0)=-\omega\epsilon_0\Delta_{\mu\alpha}(\mathbf{k},\omega)\,\mathrm{Re}\,D^{-1}_{[\alpha\beta]}(\mathbf{k},\omega,\mathbf{B}_0)\,\Delta_{\beta\nu}(\mathbf{k},\omega)\tag{61}$$

Direct insertion of (60) and (61) back into the $\hat{\sigma}$ rules immediately leads to

$$\int_0^\infty d\omega'\,\omega'\Delta_{\mu\alpha}(\mathbf{k},\omega')\,\mathrm{Im}\,D^{-1}_{(\alpha\beta)}(\mathbf{k},\omega',\mathbf{B}_0)\,\Delta_{\beta\nu}(\mathbf{k},\omega')=\tfrac{1}{2}\pi\omega_0{}^2\,\delta_{\mu\nu}\tag{62}$$

$$\int_0^\infty d\omega'\,\Delta_{\mu\alpha}(\mathbf{k},\omega')\,\mathrm{Re}\,D^{-1}_{[\alpha\beta]}(\mathbf{k},\omega',\mathbf{B}_0)\,\Delta_{\beta\nu}(\mathbf{k},\omega')=-(\pi/2\epsilon_0)\,\hat{\sigma}_{[\mu\nu]}(\mathbf{k},\omega=0,\mathbf{B}_0)\tag{63}$$

$$\int_0^\infty d\omega'\,\omega'^2\Delta_{\mu\alpha}(k,\omega')\,\mathrm{Re}\,D^{-1}_{[\alpha\beta]}(\mathbf{k},\omega',\mathbf{B}_0)\,\Delta_{\beta\nu}(k,\omega')=\tfrac{1}{2}\pi\omega_0{}^2\omega_c\,\epsilon_{\mu\nu\lambda}\hat{B}_{0\lambda}\tag{64}$$

In particular, premultiplication and postmultiplication of (62) by $\mathbf{k}$ yields

$$\int_0^\infty d\omega'\,\omega'\mathbf{k}\cdot[\mathrm{Im}\,\mathsf{D}^{-1}(\mathbf{k},\omega',\mathbf{B}_0)]\cdot\mathbf{k}=\tfrac{1}{2}\pi\omega_0{}^2\tag{65}$$

[16] See Ref. 4a, p. 347, Eq. (84.10); Ref. 7a, p. 136, Eq. (3.138); Ref. 7b, p. 205, Eq. (4.11); Ref. 23, pp. 39, 61, 242; Ref. 24.

When there is no external magnetic field acting ($\mathbf{B}_0 = 0$), Eq. (65) becomes the well-known longitudinal f-sum rule (see footnote 16):

$$\int_0^\infty d\omega'\, \omega'\, \mathrm{Im}[1/\epsilon^{LL}(\mathbf{k}, \omega')] = -\tfrac{1}{2}\pi\omega_0^2 \tag{66}$$

while (62) splits, giving (66) and the transverse rule

$$\int_0^\infty \frac{d\omega'}{\omega'} (k^2c^2 - \omega'^2)\, \mathrm{Im}\, \frac{\alpha^{TT}(\mathbf{k}, \omega')}{D^{TT}(\mathbf{k}, \omega')} = \frac{\pi\omega_0^2}{2} \tag{67}$$

Equation (66) can also be formulated directly from the Kramers–Kronig relation (43a) using the high-frequency ($\omega \to \infty$) denominator expansion technique described earlier for $\boldsymbol{\alpha}$. Or, if one starts from the Kramers–Kronig relation (38a), the denominator expansion method yields both (66) and

$$\int_0^\infty \frac{d\omega'}{\omega'}\, \mathrm{Im}\, \frac{1}{\epsilon^{LL}(\mathbf{k}, \omega')} = -\frac{\pi}{2}\, \mathrm{Re}\left[1 - \frac{1}{\epsilon^{LL}(\mathbf{k}, 0)}\right] \tag{68}$$

[see Ref. 7b, p. 210, Eq. (4.30)]. In particular, when $\mathbf{k}$ tends to zero, use of the Vlasov expression $\epsilon^{LL}(\mathbf{k}, 0) = 1 + (\kappa^2/k^2)$ in (67) leads to the long-wavelength "perfect-screening" sum rule

$$\lim_{\mathbf{k}\to 0} \int_0^\infty \frac{d\omega'}{\omega'}\, \mathrm{Im}\, \frac{1}{\epsilon^{LL}(\mathbf{k}, \omega')} = -\frac{\pi}{2} \tag{69}$$

[see Ref. 7b, p. 210, Eq. (4.31)]. This derivation suggests that (69) is valid only for *warm, collisionless plasmas*, although its domain of validity might be more extended. It is interesting to observe that (68) can be alternatively derived from (43a) by setting $\omega = 0$ therein.

Turning next to the corresponding transverse rules, if one sets $\omega = 0$ in (44a), there results

$$\int_0^\infty \frac{d\omega'}{\omega'}\, \mathrm{Im}\, \frac{\alpha^{TT}(\mathbf{k}, \omega')}{D^{TT}(\mathbf{k}, \omega')} = 0 \tag{70}$$

Hence, (67) simplifies to

$$\int_0^\infty d\omega'\, \omega'\, \mathrm{Im}\, \frac{\alpha^{TT}(\mathbf{k}, \omega')}{D^{TT}(\mathbf{k}, \omega')} = -\frac{\pi\omega_0^2}{2} \tag{71}$$

It is not surprising to see that this last result can also be derived from (44a) by the high-frequency denominator-expansion technique. To summarize, we present the longitudinal and transverse f-sum rules in Table IV.

It is now clear that one can extract a great deal from Maxwell's equations and the linear causal constitutive relations with minimal recourse to particle dynamics. However, any extensions of the theory of electrodynamic transport coefficients presented thus far must necessarily involve some suitable description of the particle

Table IV. *f*-Sum Rules

n	$(2/\pi)\int_0^\infty d\omega\, \omega^n\, \mathrm{Im}[1/\epsilon^{LL}(\omega, \mathbf{k})]$	$(2/\pi)\int_0^\infty d\omega\, \omega^n\, \mathrm{Im}[\alpha^{TT}(\mathbf{k}, \omega)/D^{TT}(\mathbf{k}, \omega)]$
-1	$-\mathrm{Re}\{1 - [1/\epsilon^{LL}(\mathbf{k}, 0)]\}$	0
$+1$	$-\omega_0^2$	$-\omega_0^2$

dynamics. In Section 3, we present such a description starting from the statistical-mechanical Liouville equation.

Relativistic Sum Rules. Particle dynamics enter the sum-rule derivations only through the simple cold-plasma relatives, determining the coefficients in the asymptotic expansion. Hence, relativistic effects appear as modifications of these coefficients. So far as the diagonal sum rules are concerned, merely the plasma frequency is of interest. Simple manipulation with the relativistic equation of motion reveals that the appropriate expression for the plasma frequency becomes[17]

$$\omega_{\text{plasma}}^2 = \omega_0^2 \left\langle \gamma^{-1}\left(1 - \frac{v^2}{3c^2}\right)\right\rangle \qquad \gamma^{-1} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{72}$$

For the off-diagonal elements, relativistic equivalents of combinations involving the plasma frequency and the cyclotron frequency are needed. One can derive these either by a high-frequency expansion of the appropriate relativistic expression for the polarizability in the Vlasov approximation (which is exact in the $\omega \to \infty$ limit) given by Trubnikov,[25] or through the equation of motion, which becomes, however, somewhat unwidely. The appropriate replacement turns out to be

$$\omega_c\omega_0^2 \to \omega_c\omega_0^2 \left\langle \gamma^{-2}\left(1 - \frac{2v^2}{3c^2}\right)\right\rangle \tag{73}$$

[17] Our simple equation-of-motion result is corroborated by considering the $\omega \to \infty$ limit of the relativistic Vlasov polarizability[31]

$$\alpha(\mathbf{k}, \omega) = -\frac{\omega_0^2}{k^2} m \int \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{p}}{\omega + \mathbf{k}\cdot\mathbf{v}}\, d^3\mathbf{p}$$

for the magnetic field-free ($\mathbf{B}_0 = 0$) electron plasma, where $\mathbf{p}$ is the momentum and F is the relativistic Maxwellian distribution function normalized so that $\int d^3\mathbf{p}\, F = 1$. Passing to this limit, we then have

$$\begin{aligned}\alpha(\mathbf{k}, \omega\to\infty) &\to \frac{\omega_0^2 m}{\omega^2 k^2} k_\alpha k_\beta \int d^3\mathbf{p}\, v_\alpha \frac{\partial F}{\partial p_\beta} = -\frac{\omega_0^2 m}{\omega^2 k^2} k_\alpha k_\beta \int d^3\mathbf{p}\, F \frac{\partial v_\alpha}{\partial p_\beta} \\ &= -\frac{\omega_0^2}{\omega^2} k_\alpha k_\beta \int d^3\mathbf{p}\, F\gamma^{-1}\left(\delta_{\alpha\beta} - \frac{v_\alpha v_\beta}{c^2}\right) \\ &= -\frac{\omega_0^2}{\omega^2}\left\langle \gamma^{-1}\left(1 - \frac{v^2}{3c^2}\right)\right\rangle^0\end{aligned}$$

3. STATISTICAL MECHANICS OF RESPONSE FUNCTIONS

Section 3 is divided into five subsections. In the first three, we present several principal formulations of the *equilibrium* fluctuation-dissipation theorem for non relativistic classical plasmas with and without constant external magnetic fields. The equilibrium state of the stationary and homogenous system of plasma particles and electromagnetic field will be suitably represented by a macrocanonical distribution [cf. Eq. (76) below]. Our treatment follows the statistical-mechanical method of Kubo[2c,13–15] in establishing the relationship between the external conductivity and the current-density fluctuation spectrum. We shall see that, while the Newtonian microscopic equations of motion for the relativistic and nonrelativistic cases differ, the fluctuation-dissipation theorems for these cases are nevertheless the same. In Sections 3.4 and 3.5, certain symmetry relations and higher-order sum rules for the transport coefficients are developed from the FDT.

The essence of the statistical-mechanical method of Kubo is this: Starting from the formal solution of the Liouville equation perturbed from the initial state of equilibrium, one calculates the response of the system to a small external perturbing agency, specified here to be the vector potential (see footnote 3)

$$\hat{\mathbf{A}}(\mathbf{r}, t) = (1/L^3)\hat{\mathbf{A}}_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{r}) \tag{74}$$

Since this response is, in turn, connected to $\partial\hat{\mathbf{A}}/\partial t$ by the external conductivity [see Eq. (14)], one is led to a relationship between the zero-order current correlation and first-order conductivity tensors (with respect to the perturbation).

The Liouville distribution function Ω is normalized to unity, so that the ensemble-averaged current density is given by

$$\langle\mathbf{j}\rangle = \int d\mathbf{\Gamma}^R\, \Omega\mathbf{j} \tag{75}$$

where $d\mathbf{\Gamma}^R$ is an element of hypervolume in the Γ-phase space spanned by the coordinates and momenta of the particles and field[16] (the precise significance of the superscript R will be discussed below).

3.1. The Nonrelativistic FDT for Electron Plasmas

Description of the Unperturbed System. Let us first consider a plasma model in which only the N electrons (each carrying charge $e = -|e|$) confined in a large spatial volume L^3 play the dynamical role; the positive ions serve to provide a uniformly smeared out background. Such a model is hereafter referred to as an *electron plasma.*

We take Ω^0 to represent the macrocanonical distribution

$$\Omega^0 = Z^{-1}\exp(-\beta H^0), \qquad Z = \int d\mathbf{\Gamma}^R \exp(-\beta H^0) \tag{76}$$

at temperature β^{-1} (in energy units) characterizing the stationary and homogeneous state of the system in the infinite past. Adopting the gauge where the scalar potential ϕ

is set equal to zero, the nonrelativistic equilibrium Hamiltonian H^0 which includes interaction[18] is

$$H^0 = \sum_{i=1}^{N} \tfrac{1}{2}mv_i^2 + \sum_{\mathbf{l}} (\epsilon_0/2L^3)(E_{\mathbf{l}\mu}E_{-\mathbf{l}\mu} + c^2l^2T_{\mu\nu}A_{\mathbf{l}\mu}A_{-\mathbf{l}\nu}) + (B_0^2/2\mu_0)\,L^3, \tag{77}$$

where

$$\mathbf{v}_i(\mathbf{x}_i\,,\mathbf{p}_i\,,\{\mathbf{A}_\mathbf{q}\}) = (1/m)[\mathbf{p}_i - (e/L^3)\sum_{\mathbf{q}} \mathbf{A}_{-\mathbf{q}}(\exp -iq\cdot\mathbf{x}_i) - e\mathbf{A}_0(\mathbf{x}_i)] \tag{78}$$

is the velocity of the ith particle, $\mathbf{A}_0$ is the vector potential corresponding to the constant external magnetic field $\mathbf{B}_0$, and the $\mathbf{x}_i$, $\mathbf{p}_i$, $\mathbf{A}_\mathbf{q}$, and $\mathbf{E}_\mathbf{q}$ are the particle and field coordinates and momenta. Having set $\phi = 0$, we observe that the entire longitudinal responsibility is shifted to the vector potential, i.e.,

$$\mathbf{E}_\mathbf{k}(t) = -\dot{\mathbf{A}}_\mathbf{k}(t) \tag{79}$$

This has important consequences, for, on the one hand, the $\mathbf{A}_\mathbf{k}^L$, like the $\mathbf{A}_\mathbf{k}^T$, are regarded as being independent field coordinates, while, on the other hand, the longitudinal components of the $\mathbf{E}_\mathbf{k}$ are *not independent field momenta.* More precisely, the $\mathbf{E}_\mathbf{k}^L$ are necessarily constrained to obey the subsidiary conditions:

$$\mathbf{k}\cdot\mathbf{E}_\mathbf{k}(t) = -(ie/\epsilon_0)\sum_i \exp[-i\mathbf{k}\cdot\mathbf{x}_i(t)] \tag{80}$$

showing that each $\mathbf{E}_\mathbf{k}^L$ depends on all the particle coordinates. There is no such subsidiary condition for the $\mathbf{A}_\mathbf{k}^L$. Thus, regarding the Γ-space as being spanned by the $\mathbf{x}_i$, $\mathbf{p}_i$, $\mathbf{A}_\mathbf{k}$, and $\mathbf{E}_\mathbf{k}$, the infinitesimal element of *reduced* phase volume is given by

$$\begin{aligned} d\mathbf{\Gamma}^R &= d\{\mathbf{x}_i\}\,d\{\mathbf{p}_i\}\,d\{\mathbf{A}_\mathbf{k}\}\,d\{\mathbf{E}_\mathbf{k}\}\prod_\mathbf{q}\delta[\mathbf{q}\cdot\mathbf{E}_\mathbf{q} + (ie/\epsilon_0)\sum_i\exp(-i\mathbf{q}\cdot\mathbf{x}_i)] \\ &\equiv d\mathbf{\Gamma}\prod_\mathbf{q}\delta[\mathbf{q}\cdot\mathbf{E}_\mathbf{q} + (ie/\epsilon_0)\sum_i\exp(-i\mathbf{q}\cdot\mathbf{x}_i)] \end{aligned} \tag{81}$$

The microscopic charge and current densities of the unperturbed system are given by

$$\rho^0(\mathbf{r}, t) = e\sum_i \delta(\mathbf{r} - \mathbf{x}_i(t)) \tag{82}$$

$$\mathbf{j}^0(\mathbf{r}, t) = e\sum_i \mathbf{v}_i\delta(\mathbf{r} - \mathbf{x}_i(t)) \tag{83}$$

with spatial Fourier transforms

$$\rho_\mathbf{k}^0(t) = e\sum_i \exp[-i\mathbf{k}\cdot\mathbf{x}_i(t)] \tag{84}$$

$$\mathbf{j}_\mathbf{k}^0(t) = e\sum_i \mathbf{v}_i\exp[-i\mathbf{k}\cdot\mathbf{x}_i(t)] \tag{85}$$

[18] It should be clear that the longitudinal interaction in the $\phi = 0$ gauge is portrayed by the terms

$$\frac{e}{mL^3}\sum_i\sum_\mathbf{q}[\mathbf{p}_i^L - e\mathbf{A}_0^L(\mathbf{x}_i)]\cdot\mathbf{A}_{-\mathbf{q}}^L\exp -i\mathbf{q}\cdot\mathbf{x}_i\,, \qquad \frac{\epsilon_0}{2L^3}\sum_\mathbf{q}\mathbf{E}_\mathbf{q}^L\cdot\mathbf{E}_{-\mathbf{q}}^L$$

In the sequel, we focus our attention on the charge and current correlations, taken at two different space-time points, and averaged over the equilibrium ensemble. Letting

$$\langle\cdots\rangle^0 = \int d\Gamma^R\, \Omega^0(\cdots)$$

denote the expectation value of any quantity over the equilibrium ensemble, these correlations are

$$P(\xi, \tau, \mathbf{B}_0) = \langle\rho^0(\mathbf{r}, t)\rho^0(\mathbf{r} + \xi, t + \tau)\rangle^0 = \langle\rho^0(0, 0)\rho^0(\xi, \tau)\rangle^0 \tag{86}$$

$$Q_{\mu\nu}(\xi, \tau, \mathbf{B}_0) = \langle j_\mu{}^0(\mathbf{r}, t) j_\nu{}^0(\mathbf{r} + \xi, t + \tau)\rangle^0 = \langle j_\mu{}^0(0, 0) j_\nu{}^0(\xi, \tau)\rangle^0 \tag{87}$$

and, for the stationary and homogeneous system, the corresponding Fourier transforms become

$$P(\mathbf{k}, \tau, \mathbf{B}_0) = (1/L^3)\langle\rho_{\mathbf{k}}{}^0(\tau)\, \rho^0_{-\mathbf{k}}(0)\rangle^0 \tag{88}$$

$$Q_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0) = (1/L^3)\langle j^0_{\mathbf{k}\nu}(\tau)\, j^0_{-\mathbf{k}\mu}(0)\rangle^0 \tag{89}$$

$$P(\mathbf{k}, \omega, \mathbf{B}_0)\, \delta(\omega - \omega') = (1/2\pi L^3)\langle\rho_{\mathbf{k}}{}^0(\omega)\, \rho^0_{-\mathbf{k}}(-\omega')\rangle^0 \tag{90}$$

$$Q_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0)\, \delta(\omega - \omega') = (1/2\pi L^3)\langle j^0_{\mathbf{k}\nu}(\omega)\, j^0_{-\mathbf{k}\mu}(-\omega')\rangle^0 \tag{91}$$

Making use of the continuity equation

$$\omega\rho_{\mathbf{k}}{}^0(\omega) = \mathbf{k}\cdot\mathbf{j}_{\mathbf{k}}{}^0(\omega) \tag{92}$$

one then obtains the following relationship between the charge and current density fluctuation spectra:

$$\omega^2 P(\mathbf{k}, \omega, \mathbf{B}_0) = \mathbf{k}\cdot\mathsf{Q}(\mathbf{k}, \omega, \mathbf{B}_0)\cdot\mathbf{k} \tag{93}$$

Further on, we shall encounter the equal-time current and charge density correlation functions $\mathsf{Q}(\mathbf{k}, \tau = 0, \mathbf{B}_0)$ and $P(\mathbf{k}, \tau = 0, \mathbf{B}_0)$, so that it is useful to present their calculations at this time. From (89) and (88), one readily obtains

$$\begin{aligned} Q_{\mu\nu}(\mathbf{k}, \tau = 0, \mathbf{B}_0) &= (1/L^3)\sum_{i,j} e^2\langle v_{i\nu}v_{j\mu}\rangle^0\, \langle\exp[-ik\cdot(\mathbf{x}_i - \mathbf{x}_j)]\rangle^0 \\ &= n_0 e^2\delta_{\mu\nu}\tfrac{1}{3}\langle v^2\rangle^0 \equiv (\epsilon_0\omega_0{}^2/\beta)\,\delta_{\mu\nu} \end{aligned} \tag{94}$$

where n_0 and ω_0 are the equilibrium values of the number density and plasma frequency.[19] Equation (94) reflects the fact that velocities are uncorrelated in the equi-

[19] We note that Eq. (94) can be alternatively derived via the formal steps:

$$\frac{1}{L^3}\sum_{i,j} e^2\,\langle v_{i\nu}(0)v_j(0)\exp -i\mathbf{k}\cdot[\mathbf{x}_i(0) - \mathbf{x}_j(0)]\rangle^0$$

$$= \frac{e^2}{L^3}\sum_{i,j}\left\langle v_{j\mu}\,[\exp -i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)]\,\frac{\partial H^0}{\partial p_{i\nu}}\right\rangle^0$$

librium ensemble. For the calculation of the equal-time density correlation function, it is necessary to introduce the one- and two-particle equilibrium distribution functions, $F(\mathbf{x}, \mathbf{p})$ and

$$G(\mathbf{x}_1, \mathbf{p}_1, \mathbf{x}_2, \mathbf{p}_2) = G(12) = N(N-1) \int d^3\mathbf{x}_3 \cdots d^3\mathbf{x}_N\, d^3\mathbf{p}_3 \cdots d^3\mathbf{p}_N \cdots d^3\mathbf{A}_1 \cdots d^3\mathbf{E}_1 \cdots \Omega^0 \tag{95}$$

and the pair correlation function g,[26] defined by the equation

$$G(12) = F(1)\, F(2)[1 + g(|\, \mathbf{x}_2 - \mathbf{x}_1 \,|)] \tag{96}$$

where $F(1)$ and $F(2)$ are normalized so that $\int F(i)\, d^3\mathbf{p}_i = n_0$. Then, evaluating the electronic charge density correlation P_e first, one finds that

$$\begin{aligned} P_e(\mathbf{k}, \tau = 0, \mathbf{B}_0) &= \frac{e^2}{L^3} \sum_{i,j} \langle \exp\{-i\mathbf{k} \cdot [\mathbf{x}_i(0) - \mathbf{x}_j(0)]\}\rangle^0 \\ &= \frac{e^2}{L^3} N(N-1) \langle \exp\{-i\mathbf{k} \cdot [\mathbf{x}_1(0) - \mathbf{x}_2(0)]\}\rangle^0 + e^2 n_0 \\ &= \frac{e^2}{L^3} \int\!\!\int\!\!\int\!\!\int d^3\mathbf{x}_1\, d^3\mathbf{x}_2\, d^3\mathbf{p}_1\, d^3\mathbf{p}_2 \{\exp[-i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)]\}\, G(12) + e^2 n_0 \\ &= \frac{e^2 n_0^{\,2}}{L^3} \int\!\!\int d^3\mathbf{x}_1\, d^3\mathbf{x}_2 \{\exp[-i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)]\}[1 + g(|\, \mathbf{x}_2 - \mathbf{x}_1 \,|)] + e^2 n_0 \\ &= e^2 n_0 [1 + n_0 g(k)] + e^2 n_0^{\,2} L^3 \delta_{\mathbf{k},0} \end{aligned} \tag{97}$$

Since the uniform static ion background contributes the compensating term $-e^2 n_0^{\,2} L^3 \delta_{\mathbf{k},0}$, it follows that

$$P(\mathbf{k}, \tau = 0, \mathbf{B}_0) = e^2 n_0 [1 + n_0 g(k)] \tag{98}$$

for the electron plasma.

Linear Response Theory. We now turn to analyze the effect of and response to a small perturbation which removes the system from equilibrium.

$$\begin{aligned} &= -\frac{e^2}{\beta L^3} \sum_{i,j} \int d\Gamma^R\, v_{j\mu} \frac{\partial}{\partial p_{i\nu}} \{\Omega^0 \exp -i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)\} \\ &= \frac{e^2}{\beta L^3} \sum_{i,j} \int d\Gamma^R\, \Omega^0 [\exp -i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)] \frac{\partial v_{j\mu}}{\partial p_{i\nu}} = \frac{e^2}{\beta L^3} \sum_{i=j} \int d\Gamma^R\, \Omega^0 \frac{\partial v_{i\nu}}{\partial p_{i\nu}} \\ &= \frac{n_0 e^2}{\beta m} \delta_{\mu\nu} \equiv \frac{\epsilon_0 \omega_0^{\,2}}{\beta} \delta_{\mu\nu} \end{aligned}$$

[cf. Eq. (78)].

The introduction of the small external agency $\hat{\mathbf{A}}$ into the system produces the perturbation expansions

$$H = H^0 + H' \tag{99}$$

$$\Omega = \Omega^0 + \Omega' \tag{100}$$

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{j}^0(\mathbf{r}, t) + \mathbf{j}'(\mathbf{r}, t) \tag{101}$$

where

$$H' = -(1/L^3)\, \hat{A}_{\mathbf{k}\mu}(t) j^0_{-\mathbf{k}\mu}(t) \tag{102}$$

is the Hamiltonian for the interaction between the system and $\hat{\mathbf{A}}$, and

$$\mathbf{j}'(\mathbf{r}, t) = -(e^2/m) \sum_i \hat{\mathbf{A}}(\mathbf{x}_i, t)\, \delta(\mathbf{r} - \mathbf{x}_i(t)) \tag{103}$$

is the small change in the microscopic current density due to the presence of $\hat{\mathbf{A}}$. Then, upon noting that $\Omega'(\mathbf{\Gamma}, t = -\infty) = 0$ (the perturbation is turned on adiabatically), the subsequent perturbation of the Liouville equation

$$\partial\Omega(\mathbf{\Gamma}, t)/\partial t = -i\mathscr{L}(\mathbf{\Gamma})\Omega \tag{104}$$

where

$$\mathscr{L} = -i[H, \ldots] \equiv -i \sum_i \left(\frac{\partial H}{\partial \mathbf{p}_i} \cdot \frac{\partial}{\partial \mathbf{x}_i} - \frac{\partial H}{\partial \mathbf{x}_i} \cdot \frac{\partial}{\partial \mathbf{p}_i}\right) + \frac{iL^3}{\epsilon_0} \sum_{\mathbf{l}} \left(\frac{\partial H}{\partial \mathbf{E}_{-\mathbf{l}}} \cdot \frac{\partial}{\partial \mathbf{A}_{\mathbf{l}}} - \frac{\partial H}{\partial \mathbf{A}_{-\mathbf{l}}} \cdot \frac{\partial}{\partial \mathbf{E}_{\mathbf{l}}}\right) \tag{105}$$

ultimately leads to the solution

$$\Omega'(\mathbf{\Gamma}, t) = \frac{\beta}{L^3} \int_0^\infty d\tau\, \hat{A}_{\mathbf{k}\mu}(t - \tau)[\exp(-i\tau\mathscr{L}^0)]\, \Omega^0 \frac{d}{dt} j^0_{-\mathbf{k}\mu}(t) \tag{106}$$

The microscopic current density depends on time implicitly through the particle momenta and the particle and field coordinates. Then, clearly, the dynamical variable $j^0_{-\mathbf{k}\mu}(\mathbf{\Gamma}) = j^0_{-\mathbf{k}\mu}(\mathbf{\Gamma}(0), t)$ satisfies the *Heisenberg* euqation

$$\frac{d}{dt} j^0_{-\mathbf{k}\mu}(t - \tau) = [\exp(-i\tau\mathscr{L}^0)] \frac{d}{dt} j^0_{-\mathbf{k}\mu}(t) \tag{107}$$

where it is understood that $(t - \tau)$ refers to time displacement determined by the dynamics of the unperturbed system. Putting (107) back into (106), one finally obtains

$$\Omega'(\mathbf{\Gamma}, t) = \frac{\beta\Omega^0}{L^3} \int_0^\infty d\tau\, \hat{A}_{\mathbf{k}\mu}(t - \tau) \frac{d}{dt} j^0_{-\mathbf{k}\mu}(t - \tau) \tag{108}$$

Now we calculate the expectation value of the current in the perturbed ensemble which, in the *Schrödinger* picture, is

$$
\begin{aligned}
\langle \mathbf{j} \rangle &= \int d\mathbf{\Gamma}^R\, \mathbf{j}(\mathbf{\Gamma})\, \Omega(\mathbf{\Gamma}, t) \\
&= \int d\mathbf{\Gamma}^R\, \mathbf{j}(\mathbf{\Gamma})\, \Omega^0 + \int d\mathbf{\Gamma}^R\, \mathbf{j}(\mathbf{\Gamma})\, \Omega'(\mathbf{\Gamma}, t) \\
&= \langle \mathbf{j} \rangle^0 + \langle \mathbf{j} \rangle' = \langle \mathbf{j}' \rangle^0 + \langle \mathbf{j}^0 \rangle'
\end{aligned} \tag{109}
$$

since $\langle \mathbf{j}^0 \rangle^0 = 0$ and $\langle \mathbf{j}' \rangle'$ is of second order of smallness. First, we calculate $\langle \mathbf{j}^0 \rangle'$. From (108) and (109),

$$
\begin{aligned}
\langle j^0_{\mathbf{k}\nu}(t) \rangle' &= \int d\mathbf{\Gamma}^R\, j^0_{\mathbf{k}\nu}(\mathbf{\Gamma})\, \Omega'(\mathbf{\Gamma}, t) \\
&= \frac{\beta}{L^3} \int_0^\infty d\tau\, \hat{A}_{\mathbf{k}\mu}(t-\tau) \int d\mathbf{\Gamma}^R\, \Omega^0 j^0_{\mathbf{k}\nu}(t) \frac{d}{dt} j^0_{-\mathbf{k}\mu}(t-\tau) \\
&= \frac{-\beta}{L^3} \int_0^\infty d\tau\, \hat{A}_{\mathbf{k}\mu}(t-\tau) \frac{d}{d\tau} \langle j^0_{\mathbf{k}\nu}(t)\, j^0_{-\mathbf{k}\mu}(t-\tau) \rangle^0 \\
&= -\beta \int_0^\infty d\tau\, \hat{A}_{\mathbf{k}\mu}(t-\tau) \frac{d}{d\tau} Q_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0) \\
&= \beta \int_0^\infty d\tau\, \hat{E}_{\mathbf{k}\mu}(t-\tau)\, Q_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0) + \beta A_{\mathbf{k}\mu}(t)\, Q_{\mu\nu}(\mathbf{k}, \tau = 0, \mathbf{B}_0)
\end{aligned} \tag{110}
$$

Here, we have exploited the fact that the stationary system is invariant with respect to a temporal translation by $(\tau - t)$. Next from (74), (103), and (109),

$$
\begin{aligned}
\langle j'_{\mathbf{l}\nu}(t) \rangle^0 &= \int d\mathbf{\Gamma}^R\, j'_{\mathbf{l}\nu}(\mathbf{\Gamma})\, \Omega^0 = -\frac{e^2}{mL^3} \hat{A}_{\mathbf{k}\nu}(t) \sum_i \int d\mathbf{\Gamma}^R\, \Omega^0 \exp[i\mathbf{x}_i \cdot (\mathbf{k} - \mathbf{l})] \\
&= -\frac{e^2}{mL^3} \hat{A}_{\mathbf{k}\nu}(t)\, \delta_{\mathbf{k},\mathbf{l}} \sum_i \int d\mathbf{\Gamma}^R\, \Omega^0 = -\epsilon_0 \omega_0{}^2 \hat{A}_{\mathbf{k}\nu}(t)\, \delta_{\mathbf{k},\mathbf{l}}
\end{aligned}
$$

so that

$$
\langle j'_{\mathbf{k}\nu}(t) \rangle^0 = -\epsilon_0 \omega_0{}^2 \hat{A}_{\mathbf{k}\nu}(t) \tag{111}
$$

Then, combining (109)–(111) and taking account of (94), one obtains

$$
\langle j_{\mathbf{k}\nu} \rangle(t) = \beta \int_0^\infty d\tau\, \hat{E}_{\mathbf{k}\mu}(t-\tau)\, Q_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0) \tag{112}
$$

In contrast to the transport coefficients, $Q_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0)$ does not obey any causality requirement. Since taking the positive time projection in the τ-domain is tantamount to a plus projection in the ω-domain [cf. Eq. (30)], we have

$$
\langle j_{\mathbf{k}\nu}(\omega) \rangle = \beta \hat{E}_{\mathbf{k}\mu}(\omega) \int_{-\infty}^{\infty} d\omega'\, \delta_+(\omega - \omega')\, Q_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0) \tag{113}
$$

Fluctuation-Dissipation Theorems. Equation (113) and Ohm's law

$$\langle j_{\mathbf{k}\nu}(\omega)\rangle = \hat{\sigma}_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0)\, \hat{E}_{\mathbf{k}\mu}(\omega)$$

can then be contracted into the first and most concise statement of the FDT[20]

$$\hat{\sigma}_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0) = \beta \int_{-\infty}^{\infty} d\omega'\, \delta_+(\omega - \omega')\, Q_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0) \tag{114}$$

A second, physically more illuminating form can be derived by focusing our attention on the dissipative part of $\hat{\sigma}$. First,observe from (91) and the reality condition that Q is a Hermitian object—namely,

$$\mathsf{Q}(\mathbf{k}, \omega, \mathbf{B}_0) = \mathsf{Q}^\dagger(\mathbf{k}, \omega, \mathbf{B}_0) \tag{115}$$

Hence,

$$\begin{aligned} \hat{\sigma}^*_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) &= \beta \int_{-\infty}^{\infty} d\omega'\, \delta_-(\omega - \omega')\, Q^*_{\nu\mu}(\mathbf{k}, \omega', \mathbf{B}_0) \\ &= \beta \int_{-\infty}^{\infty} d\omega'\, \delta_-(\omega - \omega')\, Q_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0) \end{aligned} \tag{116}$$

Then, upon combining (114) and (116), one obtains the second form

$$\hat{\sigma}^{\vee}_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0) = \tfrac{1}{2}\beta Q_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) \tag{117}$$

Equation (117), in turn, permits us to eliminate the inconvenient $\hat{\sigma}$ and to formulate the FDT in terms of $\mathsf{D}^{-1\wedge}$. Starting from (59), we have

$$\hat{\sigma}^\dagger(\mathbf{k}, \omega, \mathbf{B}_0) = i\omega\epsilon_0 \Delta(\mathbf{k}, \omega) \cdot \mathsf{D}^{-1\dagger}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \Delta(\mathbf{k}, \omega) - i\omega\epsilon_0 \Delta(\mathbf{k}, \omega) \tag{118}$$

Addition of (59) and (118) then gives

$$\hat{\sigma}^{\vee}(\mathbf{k}, \omega, \mathbf{B}_0) = -i\omega\epsilon_0 \Delta(\mathbf{k}, \omega) \cdot \mathsf{D}^{-1\wedge}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \Delta(\mathbf{k}, \omega) \tag{119}$$

which, together with (117) yields the third statement of the FDT:

$$D^{-1\wedge}_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) = (i\beta/2\omega\epsilon_0)\, \Delta^{-1}_{\nu\alpha}(\mathbf{k}, \omega)\, Q_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{B}_0)\, \Delta^{-1}_{\beta\mu}(\mathbf{k}, \omega) \tag{120}$$

Now, the behavior of Q depends on the behavior of correlations in the system. A knowledge of Q necessarily involves a knowledge of two-time correlation functions, which would, in principle, have to be determined from some type of kinetic equation. Therefore, it seems to be easier to express Q in terms of D, and, hence, one salient advantage of (120). The longitudinal form of (120) is

$$\mathbf{k} \cdot \mathsf{D}^{-1\wedge}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{k} = (i\beta\omega/2\epsilon_0 k^2)\, P(\mathbf{k}, \omega, \mathbf{B}_0) \tag{121}$$

[see Ref. 19, p. 117, Eq. (14.23)]. It is interesting to remark briefly on the third statement of the FDT when there is no external magnetic field acting ($\mathbf{B}_0 = 0$). In this case,

[20] See, for example, Ref. 2c, p. 580, Eq. (5.11) with $\hbar = 0$, or Ref. 13a, p. 148, Eq. (2.54) with $\hbar = 0$; Ref. 13b, p. 282, Eq. (11.42).

Eq. (120) splits into the independent longitudinal (LL)[(8) 21] and transverse (TT) relations[22]

$$-\operatorname{Im}\frac{1}{\epsilon^{LL}(\mathbf{k},\omega)}=\operatorname{Im}\frac{\alpha^{LL}(\mathbf{k},\omega)}{\epsilon^{LL}(\mathbf{k},\omega)}=\frac{\beta\omega}{2\epsilon_0 k^2}P(\mathbf{k},\omega) \tag{122}$$

$$\operatorname{Im}\frac{\alpha^{TT}(\mathbf{k},\omega)}{D^{TT}(\mathbf{k},\omega)}=\frac{\beta\omega}{2\epsilon_0}\frac{Q^{TT}(\mathbf{k},\omega)}{k^2c^2-\omega^2} \tag{123}$$

The fourth form of the FDT relates the pair correlation function $g(r)$, which is the primary object in equilibrium statistical-mechanical calculations, to the static value of the longitudinal dielectric function. From (121) and the Kramers–Kronig formula (42),

$$\begin{aligned}P(\mathbf{k},\tau=0,\mathbf{B}_0)&=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}P(\mathbf{k},\omega',\mathbf{B}_0)\\&=\frac{-i\epsilon_0 k^2}{\pi\beta}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'}\mathbf{k}\cdot\mathsf{D}^{-1\wedge}(\mathbf{k},\omega',\mathbf{B}_0)\cdot\mathbf{k}\\&=\frac{\epsilon_0 k^2}{\beta}[1+\mathbf{k}\cdot\mathsf{D}^{-1\vee}(\mathbf{k},\omega=0,\mathbf{B}_0)\cdot\mathbf{k}]\end{aligned} \tag{124}$$

whence, elimination of P between (124) and (98) yields

$$g(k)=\frac{1}{n_0}\left\{\frac{\epsilon_0 k^2}{n_0\beta e^2}[1+\mathbf{k}\cdot\mathsf{D}^{-1\vee}(\mathbf{k},\omega=0,\mathbf{B}_0)\cdot\mathbf{k}]-1\right\} \tag{125}$$

It can easily be shown that

$$\mathbf{k}\cdot\mathsf{D}^{-1\vee}(\mathbf{k},\omega=0,\mathbf{B}_0)\cdot\mathbf{k}=-\operatorname{Re}\frac{1}{\epsilon^{LL}(\mathbf{k},\omega=0,\mathbf{B}_0)}$$

and (125) then reads

$$g(k)=\frac{1}{n_0}\left\{\frac{\epsilon_0 k^2}{n_0\beta e^2}\operatorname{Re}\left[1-\frac{1}{\epsilon^{LL}(\mathbf{k},\omega=0,\mathbf{B}_0)}\right]-1\right\} \tag{126}$$

[21] A nonequilibrium generalization of the longitudinal FDT [see Ref. 5, p. 89, Eq. (28) and Ref. 3] which, however, is more restricted in that its validity is confined to the lowest order in the coupling parameter e^2, is rendered via the idea of uncorrelated "dressed" electron density

$$\rho_{\mathbf{k}}(\omega)=\frac{2\pi e}{\epsilon^{LL}(\mathbf{k},\omega)}\sum_i\rho(\mathbf{k}\cdot\mathbf{v}_i-\omega).$$

Inserting this into our relation (90) and integrating over the arbitrary one-particle distribution function $F(v)$, one finds that

$$P(\mathbf{k},\omega)=\frac{2\pi n_0 e^2}{|\epsilon^{LL}(\mathbf{k},\omega)|^2}\int d^3\mathbf{v}\,F(\mathbf{v})\,\delta(\omega-\mathbf{k}\cdot\mathbf{v})$$

For a Maxwellian distribution, this becomes identical with our Eq. (122) in view of the Vlasov relation connecting α'' with the distribution function: $\alpha''(\mathbf{k},\omega)=(\pi m\beta\omega_0^2/k^2)(\omega/k)F(\omega/k)$, with $F(\omega/k)=(m\beta/2\pi)^{1/2}\exp(-\beta m\omega^2/2k^2)$.

[22] See Ref. 19, p. 114, Eq. (14.13) for the transverse and longitudinal FDT.

One or both of the relations (124) and (126) have been reported earlier by Akhiezer *et al*,[23] Kalman,[24] Ichimaru,[25] and Englert and Brout.[26] In particular, the first group of authors include the external magnetic field in their calculation, and it is clear that, in virtue of general statistical-mechanical arguments, the static form (126) is unaffected by the presence of $\mathbf{B}_0$.

Equation (126) can be used to derive the expression for the Debye–Hückel pair correlation function. Noting that, for the warm, collisionless plasma,

$$\epsilon^{LL}(\mathbf{k}, 0) = 1 + (\kappa^2/k^2), \qquad \kappa^2 = \beta n_0 e^2/\epsilon_0 \tag{127}$$

with or without the external magnetic field, one is led to the Debye–Hückel pair correlation function[27]

$$g(k) = \frac{-\beta e^2/\epsilon_0}{\kappa^2 + k^2}, \qquad g(r) = \frac{-\beta e^2}{\epsilon_0} \frac{e^{-\kappa r}}{r} \tag{128}$$

3.2. Nonrelativistic FDT for Ion–Electron Plasmas

Description of the Unperturbed System. Now we extend the model to include the ion as well as the electron dynamics. Let N be the number of electrons

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[23] See Ref. 25, p. 118, equation cited between (14.24) and (14.25) for the fourth ("static") form of the FDT.

[24] See Ref. 8, p. 22, Eq. (82).

[25] See Ref. 5, p. 89, Eq. (30). Remark that our $P(\mathbf{k}, t = 0)$ and Ichimaru's form factor $S(\mathbf{k})$ are the same: $P(\mathbf{k}, t = 0) = n_0 S(\mathbf{k})$.

[26] See Ref. 1c, Eq. (2.22).

[27] It is interesting to note the following: For just two test particles in equilibrium, ignoring the effect of the medium, $g(r) = e^{-\beta\phi(r)} - 1$, where r is the separation distance and $\phi(r)$ is the Coulomb potential with Fourier transform $\phi(k) = e^2/\epsilon_0 k^2$. However, the effect of the medium is such that the pair correlation function can be expressed in terms of an effective potential, $\Phi(r)$ ("potential of an effective force," see, for example, Ref. 27, pp. 481, 482, 492–495) as $g(r) = e^{-\beta\Phi(r)} - 1$, rather than in terms of the bare potential $\phi(r)$. Since the interaction is considered to be weak, we then have, approximately, $g(r) \approx -\beta\Phi(r)$, or, in Fourier language, $g(k) \approx -\beta\Phi(k)$. In general, it is difficult to determine Φ. It seems to be reasonable to assume, however, that, in some approximation, this potential is identical with the potential surrounding a test particle and its polarization cloud in a plasma. Thus, to simplify the calculation, we set $\mathbf{B}_0$ equal to zero (the results are unaltered by the presence of the external magnetic field), so that the dielectric tensor is diagonal. Considering now a test particle with charge e and constant velocity $\mathbf{v}_0$; the potential surrounding the test particle and its polarization cloud in the moving coordinate system where Φ is stationary is $\Phi(\mathbf{k}, \omega) = 2\pi\delta(\omega - \mathbf{k}\cdot\mathbf{v}_0)\,\phi(k)/\epsilon^{LL}(\mathbf{k}, \omega)$, whence

$$\Phi(\mathbf{k}) = \Phi(\mathbf{k}, t = 0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \Phi(\mathbf{k}, \omega) = \frac{\phi(\mathbf{k})}{\epsilon^{LL}(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_0)}$$

This is velocity-dependent. An obvious (although crude) approximation seems to be to set $\mathbf{v}_0$ equal to zero. This is identical to the previously derived result if, and only if,

$$\frac{1}{n_0}\left\{\frac{1}{n_0\beta\phi(k)} \operatorname{Re}\left[1 - \frac{1}{\epsilon^{LL}(\mathbf{k}, 0)}\right] - 1\right\} = \frac{-\beta\phi(\mathbf{k})}{\epsilon^{LL}(\mathbf{k}, 0)}$$

which can also be written in the form $(\kappa^4/k^4) - (\kappa^2/k^2)\,\epsilon^{LL}(\mathbf{k}, 0) + \epsilon^{LL}(\mathbf{k}, 0) - 1 = 0$, with solution $\epsilon^{LL}(\mathbf{k}, 0) = 1 + (\kappa^2/k^2)$. This shows that the identity of the exact FDT result and that of the crude static-test-particle model is a consequence of the particular algebraic form of $\epsilon^{LL}(\mathbf{k}, 0)$.

(each having charge $-|e|$ and mass m as before) and N/Z the number of ions (each having charge $+Z|e|$ and mass M). Here, the infinitesimal element of phase volume will be

$$d\Gamma = d^3\mathbf{x}_1 \cdots d^3\mathbf{x}_N \, d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_N \, d^3\mathbf{X}_1 \cdots d^3\mathbf{X}_{N/Z} \, d^3\mathbf{P}_1 \cdots d^3\mathbf{P}_{N/Z} \cdots d^3\mathbf{A}_\mathbf{q} \cdots d^3\mathbf{E}_\mathbf{q} \cdots \tag{129}$$

where $\mathbf{X}_j$ and $\mathbf{P}_j$ are the jth ion coordinate and momentum, respectively.

The stationary and homogeneous state of the system in the infinite past is again characterized by the macrocanonical distribution (76), where now of course, the ion kinetic energy $\frac{1}{2}\sum_{j=1}^{N/Z} MV_j{}^2$ must also be included in H^0 in (77).

The equilibrium microscopic electron charge and current densities are given by (82) and (83). The corresponding ion densities are

$$R^0(\mathbf{r}, t) = Z\,|\,e\,|\sum_j \delta(\mathbf{r} - \mathbf{X}_j(t)) \tag{130}$$

$$\mathbf{J}^0(\mathbf{r}, t) = Z\,|\,e\,|\sum_j \mathbf{V}_j\, \delta(\mathbf{r} - \mathbf{X}_j(t)) \tag{131}$$

These show a greater variety of relevant current correlation tensors:

$$Q_{\mu\nu}^{(ee)}(\mathbf{k}, \tau, \mathbf{B}_0) = (1/L^3)\langle j^0_{\mathbf{k}\nu}(t)\, j^0_{-\mathbf{k}\mu}(t-\tau)\rangle^0 \tag{132a}$$

$$Q_{\mu\nu}^{(ei)}(\mathbf{k}, \tau, \mathbf{B}_0) = (1/L^3)\langle j^0_{\mathbf{k}\nu}(t)\, J^0_{-\mathbf{k}\mu}(t-\tau)\rangle^0 \tag{132b}$$

$$Q_{\mu\nu}^{(ie)}(\mathbf{k}, \tau, \mathbf{B}_0) = (1/L^3)\langle J^0_{\mathbf{k}\nu}(t)\, j^0_{-\mathbf{k}\mu}(t-\tau)\rangle^0 \tag{132c}$$

$$Q_{\mu\nu}^{(ii)}(\mathbf{k}, \tau, \mathbf{B}_0) = (1/L^3)\langle J^0_{\mathbf{k}\nu}(t)\, J^0_{-\mathbf{k}\mu}(t-\tau)\rangle^0 \tag{132d}$$

while the total current correlation tensor is given by

$$Q_{\mu\nu} = Q_{\mu\nu}^{(ee)} + Q_{\mu\nu}^{(ei)} + Q_{\mu\nu}^{(ie)} + Q_{\mu\nu}^{(ii)} \tag{133}$$

The corresponding forms of the charge density correlations and of the typical relation

$$P^{(ei)}(\mathbf{k}, \omega, \mathbf{B}_0) = (1/\omega^2(\,\mathbf{k}\cdot \mathbf{Q}^{(ei)}(\mathbf{k}, \omega, \mathbf{B}_0)\cdot \mathbf{k} \tag{134}$$

are obvious.[28] Then, following the method of calculation presented in (94), one can show that

$$Q_{\mu\nu}^{(ee)}(\mathbf{k}, \tau = 0, \mathbf{B}_0) = \frac{\epsilon_0\omega_0{}^2}{\beta}\,\delta_{\mu\nu}\,, \qquad Q_{\mu\nu}^{(ii)}(\mathbf{k}, \tau = 0, \mathbf{B}_0) = Z\frac{m}{M}\frac{\epsilon_0\omega_0{}^2}{\beta}\,\delta_{\mu\nu}$$
$$Q_{\mu\nu}^{(ei)}(\mathbf{k}, \tau = 0, \mathbf{B}_0) = 0 = Q_{\mu\nu}^{(ie)}(\mathbf{k}, \tau = 0, \mathbf{B}_0) \tag{135}$$

[28] From the reality of $P^{(ei)}(\mathbf{r}, t, \mathbf{B}_0)$, microscopic time reversibility, invariance under spatial inversion, and invariance under reversal of $\mathbf{B}_0$ [P depends on $\mathbf{B}_0$ only as $|\,\mathbf{B}_0\,|$ and $(\mathbf{k}\cdot\mathbf{B}_0)^2$], one finds that $P^{(ei)}(\mathbf{k}, \omega, \mathbf{B}_0)$ is also real and that $P^{(ei)}(\mathbf{k}, \omega, \mathbf{B}_0) = P^{(ie)}(\mathbf{k}, \omega, \mathbf{B}_0)$. Hence, from (139), $g_{ei}(\xi) = g_{ie}(\xi)$.

For the calculation of the equal-time charge density correlation, it is convenient to adopt the notation

$$d1_e = d^3\mathbf{x}_1\, d^3\mathbf{p}_1\,, \qquad d1_i = d^3\mathbf{X}_1\, d^3\mathbf{P}_1$$

The two-particle distribution functions are given by

$$G(1_e\,, 2_e) = N(N-1) \int \Omega^0\, d3_e \cdots dN_e\, d1_i \cdots d\left(\frac{N}{Z}\right)_i \cdots d^3\mathbf{A}_\mathbf{q} \cdots d^3\mathbf{E}_\mathbf{q} \cdots \tag{136a}$$

$$G(1_i\,, 2_i) = \frac{N}{Z}\left(\frac{N}{Z} - 1\right) \int \Omega^0\, d3_i \cdots d\left(\frac{N}{Z}\right)_i d1_e \cdots dN_e \cdots d^3\mathbf{A}_\mathbf{q} \cdots d^3\mathbf{E}_\mathbf{q} \cdots \tag{136b}$$

$$G(1_e\,, 2_i) = N\frac{N}{Z} \int \Omega^0\, d2_e \cdots dN_e\, d1_i\, d3_i \cdots d\left(\frac{N}{Z}\right)_i \cdots d^3\mathbf{A}_\mathbf{q} \cdots d^3\mathbf{E}_\mathbf{q} \cdots \tag{136c}$$

etc. One then defines the corresponding pair correlation functions g_{ee}, g_{ii}, and g_{ei} in the same way as for the electron plasma. For example,

$$G(1_e\,, 2_i) = F_{\mathrm{el}}(1_e)\, F_{\mathrm{ion}}(2_i)[1 + g_{ei}(|\,\mathbf{x}_1 - \mathbf{X}_2\,|)], \tag{137}$$

when F_{el} and F_{ion} are equilibrium electron and ion one-particle distribution functions normalized so that

$$\int F_{\mathrm{el}}\, d^3\mathbf{p} = n_0\,, \tag{138a}$$

$$\int F_{\mathrm{ion}}\, d^3\mathbf{P} = n_0/Z \tag{138b}$$

Equations (136)–(138) then permit us to evaluate the equal-time density correlations in terms of the pair correlation functions. The results are

$$P^{(mn)}(\boldsymbol{\xi}, \tau = 0, \mathbf{B}_0) = e_m e_n n_0^2[1 + g_{mn}(\xi)] + Z_{mn} e^2 n_0\, \delta(\xi) \tag{139}$$

where $m = e, i$; $n = e, i$; $e_e = -|\,e\,|$, $e_i = |\,e\,|$; $Z_{ee} = 1$, $Z_{ii} = Z$, $Z_{ei} = 0$.

Linear Response Theory. Now, the introduction of the small external agency $\hat{\mathbf{A}}$ into the system produces the perturbation expansions (99)–(101) and

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}^0(\mathbf{r}, t) - \frac{Z^2e^2}{M} \sum_{j=1}^{N/Z} \hat{\mathbf{A}}(\mathbf{X}_j\,, t)\, \delta(\mathbf{r} - \mathbf{X}_j(t)) \tag{140}$$

where

$$H' = -(1/L^3)\, \hat{A}_{\mathbf{k}\mu}(t)[j^0_{-\mathbf{k}\mu}(t) + J^0_{-\mathbf{k}\mu}(t)] \tag{141}$$

Then, upon following the method of calculation presented in Section 3.1, one finds that

$$\Omega'(\boldsymbol{\Gamma}, t) = \frac{\beta\Omega^0}{L^3} \int_0^\infty d\tau\, \hat{A}_{\mathbf{k}\mu}(t - \tau) \frac{d}{dt} [j^0_{-\mathbf{k}\mu}(t - \tau) + J^0_{-\mathbf{k}\mu}(t - \tau)] \tag{142}$$

which, together with (109) and (135), leads to the result

$$\langle j^0_{\mathbf{k}\nu}(t)\rangle' = \beta \int_0^\infty d\tau\, \hat{E}_{\mathbf{k}\mu}(t-\tau)[Q^{(ee)}_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0) + Q^{(ei)}_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0)] + \epsilon_0 {\omega_0}^2 \hat{A}_{\mathbf{k}\nu}(t) \qquad (143)$$

The value of $\langle j'_{\mathbf{k}\nu}(t)\rangle^0$ is again given by (111) which, when combined with (143), yields the expectation value

$$\langle j_{\mathbf{k}\nu}(t)\rangle = \beta \int_0^\infty d\tau\, \hat{E}_{\mathbf{k}\mu}(t-\tau)[Q^{(ee)}_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0) + Q^{(ei)}_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0)] \qquad (144a)$$

with subsequent Fourier transform

$$\langle j_{\mathbf{k}\nu}(\omega)\rangle = \beta \hat{E}_{\mathbf{k}\mu}(\omega) \int_{-\infty}^\infty d\omega'\, \delta_+(\omega-\omega')[Q^{(ee)}_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0) + Q^{(ei)}_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0)] \qquad (144b)$$

Similarly, the expectation value of the ion current density is found to be

$$\langle J_{\mathbf{k}\nu}(\omega)\rangle = \beta \hat{E}_{\mathbf{k}\mu}(\omega) \int_{-\infty}^\infty d\omega'\, \delta_+(\omega-\omega')[Q^{(ii)}_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0) + Q^{(ie)}_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0)] \qquad (145)$$

Fluctuation–Dissipation Theorems. Eqs. (144b), (145), and the corresponding Ohm's laws

$$\langle j_{\mathbf{k}\nu}(\omega)\rangle = \hat{\sigma}^{(e)}_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0)\, \hat{E}_{\mathbf{k}\mu}(\omega) \qquad (146)$$

$$\langle J_{\mathbf{k}\nu}(\omega)\rangle = \hat{\sigma}^{(i)}_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0)\, \hat{E}_{\mathbf{k}\mu}(\omega) \qquad (147)$$

can then be contracted into the first concise form of the FDT for the electron and ion external conductivities:

$$\hat{\sigma}^{(m)}_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0) = \beta \int_{-\infty}^\infty d\omega'\, \delta_+(\omega-\omega') \sum_{n=e,i} Q^{(mn)}_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0), \qquad (m = e, i) \qquad (148)$$

The important feature that now emerges is that, in contrast to the case of the electron plasma, the FDT links two transport coefficients with the three correlation functions. Therefore, the knowledge of the transport coefficients is not sufficient to determine the correlation functions, but, rather, only a certain combination of them. This feature will reappear in various guises in the sequel. Upon summing (148) over m and taking account of (133), one then obtains the corresponding FDT for the total external conductivity $\hat{\sigma} = \hat{\sigma}^{(e)} + \hat{\sigma}^{(i)}$.

The dissipative part of $\hat{\sigma}^{(m)}$ is readily obtained by adding (148) to its Hermitian conjugate:

$$\hat{\sigma}^{(m)\vee}_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0) = \frac{1}{2}\beta Q^{(mm)}_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) + \frac{1}{2}\beta Q^{(mn)\vee}_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) + \frac{i}{2\pi}\beta \mathscr{P} \int_{-\infty}^\infty \frac{d\omega'}{\omega-\omega'}\, Q^{(mn)\wedge}_{\mu\nu}(\mathbf{k}, \omega', \mathbf{B}_0),$$
$$(m = e, i;\quad n = e, i;\quad n \neq m) \qquad (149)$$

Equation (149) can also be restated in terms of the internal response function $(\mathsf{a}^{(m)} \cdot \mathsf{D}^{-1})^{\wedge}$ by recalling from (15) and (19) that

$$\hat{\sigma}^{(m)} = -i\omega\epsilon_0 \mathsf{a}^{(m)} \cdot \mathsf{D}^{-1} \cdot \Delta \tag{150}$$

where now

$$\mathsf{D} = \Delta - \mathsf{a} = \Delta - \sum_{n=e,i} \mathsf{a}^{(n)} \tag{151}$$

In particular, when there is no external magnetic field acting, it can be shown that

$$\operatorname{Im} \frac{\alpha_{LL}^{(m)}(\mathbf{k}, \omega)}{\epsilon^{LL}(\mathbf{k}, \omega)} = \frac{\beta\omega\phi(k)}{2e^2} \sum_{n=e,i} P^{(mn)}(\mathbf{k}, \omega) \tag{152}$$

$$\operatorname{Im} \frac{\alpha_{TT}^{(m)}(\mathbf{k}, \omega)}{D^{TT}(\mathbf{k}, \omega)} = \frac{\beta\omega}{2\epsilon_0(k^2c^2 - \omega^2)} \sum_{n=e,i} Q_{TT}^{(mn)}(\mathbf{k}, \omega) \tag{153}$$

$(m = e, i)$.

To derive the static FDT in the presence of an external magnetic field, we first observe that only the longitudinal projection of the symmetric part of (149) is relevant, so that this, together with (150), ultimately gives (see footnote 28)

$$\mathbf{k} \cdot [\mathsf{a}^{(m)}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)]^{\wedge} \cdot \mathbf{k} = -\frac{i\beta\omega\phi(k)}{2e^2} \sum_{n=e,i} P^{(mn)}(\mathbf{k}, \omega, \mathbf{B}_0) \qquad (m = e, i) \tag{154}$$

Equation (154) can also be written in the form

$$\begin{aligned} \sum_{n=e,i} P^{(mn)}(\mathbf{k}, \tau = 0, \mathbf{B}_0) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{n=e,i} P^{(mn)}(\mathbf{k}, \omega, \mathbf{B}_0) \\ &= \frac{ie^2}{\pi\beta\phi(k)} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \mathbf{k} \cdot [\mathsf{a}^{(m)}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)]^{\wedge} \cdot \mathbf{k} \\ &\qquad (m = e, i) \end{aligned} \tag{155}$$

The calculation in (155) can be carried further by exploiting the Kramers–Kronig relation

$$[\mathsf{a}^{(m)}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)]^{\vee} = \frac{i}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega'} [\mathsf{a}^{(m)}(\mathbf{k}, \omega', \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega', \mathbf{B}_0)]^{\wedge} \tag{156}$$

$(m = e, i)$ with $\omega = 0$. Hence,

$$\begin{aligned} \sum_{n=e,i} P^{(mn)}(\mathbf{k}, \tau = 0, \mathbf{B}_0) &= -\frac{e^2}{\beta\phi(k)} \mathbf{k} \cdot [\mathsf{a}^{(m)}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega = 0, \mathbf{B}_0)]^{\vee} \cdot \mathbf{k} \\ &= \frac{e^2}{\beta\phi(k)} \operatorname{Re} \frac{\alpha_{LL}^{(m)}(\mathbf{k}, \omega = 0, \mathbf{B}_0)}{\epsilon^{LL}(\mathbf{k}, \omega = 0, \mathbf{B}_0)} \qquad (m = e, i) \end{aligned} \tag{157}$$

Then, upon combining (157) with the spatial Fourier transform of (139), one finally obtains the static form of the FDT for the pair correlations:

$$\sum_{n=e,i} \frac{e_m e_n}{|e|^2} g_{mn}(k) = \frac{1}{n_0} \left[\frac{1}{\beta n_0 \phi(k)} \operatorname{Re} \frac{\alpha_{LL}^{(m)}(\mathbf{k}, \omega = 0, \mathbf{B}_0)}{\epsilon^{LL}(\mathbf{k}, \omega = 0, \mathbf{B}_0)} - Z_{mm} \right] \quad (m = e, i) \qquad (158)$$

with $g_{ei}(k) = g_{ie}(k)$ (see footnote 28). Obviously, the two FDT equations given by (158) are not sufficient to determine the three independent correlation functions g_{ee}, g_{ei}, g_{ii}. In order to evaluate these correlations separately, one must resort to nonequilibrium techniques (or to other statistical-mechanical approaches).[29]

3.3. The Relativistic Fluctuation-Dissipation Theorem

While the nonrelativistic and relativistic equations of motion differ markedly, one can show that the various forms of the FDT for the relativistic plasma are the same as the corresponding forms for the nonrelativistic plasma. In the model where only the electrons play the dynamical role, the unperturbed system with interaction is

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[29] The idea of the uncorrelated "dressed" electrons employed in the nonequilibrium derivation of the longitudinal FDT for the electron plasma (cf. footnote 21) can be extended to incorporate a system of uncorrelated dressed electrons and ions (Ref. 28; Ref. 5, pp. 85, 86, Eqs. (20a, c); Ref. 3a). Taking the arbitrary electron and ion distribution functions as $F_e(\mathbf{v})$ and $F_i(\mathbf{v})$, respectively, a nontrivial generalization of the relation quoted in footnote 21 will be (we take $Z = 1$)

$$P_{ee}(\mathbf{k}, \omega) = 2\pi n_0 e^2 \left| \frac{1 + \alpha_i(\mathbf{k}, \omega)}{\epsilon(\mathbf{k}, \omega)} \right|^2 \int d^3\mathbf{v}\, F_e(\mathbf{v})\, \delta(\omega - \mathbf{k} \cdot \mathbf{v})$$

$$+ 2\pi n_0 e^2 \left| \frac{\alpha_e(\mathbf{k}, \omega)}{\epsilon(\mathbf{k}, \omega)} \right|^2 \int d^3\mathbf{v}\, F_i(\mathbf{v})\, \delta(\omega - \mathbf{k} \cdot \mathbf{v})$$

$$P_{ei}(\mathbf{k}, \omega) = -2\pi n_0 e^2 \frac{[1 + \alpha_i(\mathbf{k}, \omega)]}{\epsilon(\mathbf{k}, \omega)} \frac{\alpha_i^*(\mathbf{k}, \omega)}{\epsilon^*(\mathbf{k}, \omega)} \int d^3\mathbf{v}\, F_e(\mathbf{v})\, \delta(\omega - \mathbf{k} \cdot \mathbf{v})$$

$$- 2\pi n_0 e^2 \frac{\alpha_e(\mathbf{k}, \omega)}{\epsilon(\mathbf{k}, \omega)} \frac{[1 + \alpha_e^*(\mathbf{k}, \omega)]}{\epsilon^*(\mathbf{k}, \omega)} \int d^3\mathbf{v}\, F_i(\mathbf{v})\, \delta(\omega - \mathbf{k} \cdot \mathbf{v})$$

which, when combined, yield

$$P_{ee}(\mathbf{k}, \omega) + P_{ei}(\mathbf{k}, \omega) = \frac{2\pi n_0 e^2}{|\epsilon(\mathbf{k}, \omega)|^2} \times \left\{ [1 + \alpha_i(\mathbf{k}, \omega)] \int d^3\mathbf{v}\, F_e(\mathbf{v})\, \delta(\omega - \mathbf{k} \cdot \mathbf{v}) - \alpha_e(\mathbf{k}, \omega) \int d^3\mathbf{v}\, F_i(\mathbf{v})\, \delta(\omega - \mathbf{k} \cdot \mathbf{v}) \right\}$$

In particular, for an equilibrium electron–ion plasma, this becomes

$$P_{ee}(\mathbf{k}, \omega) + P_{ei}(\mathbf{k}, \omega) = \frac{2\pi n_0 e^2}{k\, |\epsilon(\mathbf{k}, \omega)|^2} \left\{ [1 + \alpha_i(\mathbf{k}, \omega) F_e \left(\frac{\omega}{k} \right) - \alpha_e(\mathbf{k}, \omega) F_i \left(\frac{\omega}{k} \right) \right\}$$

and, in view of the relation which connects α_e'' with $F_e(\omega/k)$ and α_i'' with $F_i(\omega/k)$ (see footnote 21) in the Vlasov approximation, this last expression is identical to (152).

again characterized by the macrocanonical distribution (76) and by Eqs. (82) and (83), where now

$$H^0 = \sum_{i=1}^{N} \gamma_i mc^2 + \sum_{\mathbf{l}} \frac{\epsilon_0}{2L^3} (E_{\mathbf{l}\mu} E_{-\mathbf{l}\mu} + c^2 l^2 T_{\mu\nu} A_{\mathbf{l}\mu} A_{-\mathbf{l}\nu}) + \frac{B_0^2}{2\mu_0} L^3 \tag{159}$$

$$\gamma_i = (1 - v_i^2/c^2)^{-1/2} \tag{160}$$

$$\mathbf{v}_i(\mathbf{x}_i, \mathbf{p}_i, \{\mathbf{A}_\mathbf{q}\}) = \frac{[\mathbf{p}_i - (e/L^3) \sum_\mathbf{q} \mathbf{A}_{-\mathbf{q}}(\exp -i\mathbf{q}\cdot\mathbf{x}_i) - e\mathbf{A}_0(\mathbf{x}_i)]/m}{\{1 + [1/(mc)^2][\mathbf{p}_i - (e/L^3)\sum_\mathbf{q} \mathbf{A}_{-\mathbf{q}}(\exp -i\mathbf{q}\cdot\mathbf{x}_i) - e\mathbf{A}_0(\mathbf{x}_i)]^2\}^{1/2}} \tag{161}$$

The equal-time density correlations for the relativistic electron plasma is still given by (98), whereas, for the equal-time current correlation tensor, one finds that

$$Q_{\mu\nu}(\mathbf{k}, \tau = 0, \mathbf{B}_0) = \frac{\epsilon_0 \omega_0^2}{\beta} \delta_{\mu\nu} \left\langle \gamma^{-1} \left(1 - \frac{v^2}{3c^2}\right)\right\rangle^0 \tag{162}$$

The introduction of the small external agency $\hat{\mathbf{A}}$ then produces the perturbations (102), (108), and

$$\mathbf{v}_i' = -\frac{e}{m} \gamma_i^{-1} \left(1 - \frac{\mathbf{v}_i\mathbf{v}_i}{c^2}\right) \cdot \hat{\mathbf{A}}(\mathbf{x}_i, t) \tag{163}$$

$$\mathbf{j}'(\mathbf{r}, t) = -\frac{e^2}{m} \sum_i \gamma_i^{-1} \left(1 - \frac{\mathbf{v}_i\mathbf{v}_i}{c^2}\right) \cdot \hat{\mathbf{A}}(\mathbf{x}_i, t)\, \delta(\mathbf{r} - \mathbf{x}_i(t)) \tag{164}$$

We are now ready to calculate expectation values according to (109). From (89), (108), and (162), it can be shown that

$$\langle j^0_{\mathbf{k}\nu}(t)\rangle' = \beta \int_0^\infty d\tau\, \hat{E}_{\mathbf{k}\mu}(t - \tau)\, Q_{\mu\nu}(\mathbf{k}, \tau, \mathbf{B}_0) + \epsilon_0\omega_0^2 \hat{A}_{\mathbf{k}\mu}(t) \left\langle \gamma^{-1} \left(1 - \frac{v^2}{3c^2}\right)\right\rangle^0 \tag{165}$$

while the appropriate ensemble average of (164) is given by

$$\langle j'_{\mathbf{k}\nu}(t)\rangle^0 = -\epsilon_0\omega_0^2 \hat{A}_{\mathbf{k}\nu}(t) \left\langle \gamma^{-1} \left(1 - \frac{v^2}{3c^2}\right)\right\rangle^0 \tag{166}$$

Consequently, addition of (165) and (166) yields (112), and, hence, the Fourier transform (113) for the expectation value of the electron current. Since the subsequent FDT relations (114), (117), (120)–(123) and (126) are then formulated independently of the particle dynamics, it follows that these relations remain unaltered for the relativistic electron plasma.

Finally, it is a simple matter to pass to the more general relativistic plasma model, where the ions, as well as the electrons, are dynamical. Our analysis shows that the FDT relations (148), (149), (152)–(154), and (158) are entirely valid for the relativistic ion–electron plasma.

3.4. Symmetry Relations

Both the current correlation tensor and the dielectric tensor satisfy certain symmetry relations which are essentially consequences of spatial and temporal inversion invariance properties of the system. Such relations were first stated by Onsager,[17] and we shall now establish them using the FDT. Since the microscopic current densities obey the Newtonian equations of motion, then, upon time reversal accompanied by the inverstion of the external magnetic field (the microscopic magnetic field automatically changes its sign following time reversal), the microscopic current undergoes the transformation

$$j_\mu(t, \mathbf{B}_0) \rightarrow -j_\mu(-t, -\mathbf{B}_0)$$

which, however, leaves the current correlation tensor invariant, i.e.,

$$Q_{\mu\nu}(\mathbf{r}, t, \mathbf{B}_0) = Q_{\mu\nu}(\mathbf{r}, -t, -\mathbf{B}_0) \tag{167}$$

Let us first consider the case of electron plasmas. From (167) and the reality of $Q_{\mu\nu}(\mathbf{r}, t, \mathbf{B}_0)$,

$$Q_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) = Q_{\mu\nu}(\mathbf{k}, -\omega, -\mathbf{B}_0) = Q^*_{\mu\nu}(-\mathbf{k}, \omega, -\mathbf{B}_0) \tag{168}$$

Recalling next the Hermiticity of Q [see Eq. (115)] one can write

$$Q^*_{\mu\nu}(-\mathbf{k}, \omega, -\mathbf{B}_0) = Q_{\nu\mu}(-\mathbf{k}, \omega, -\mathbf{B}_0) \tag{169}$$

so that contraction of (168) and (169) yields

$$Q_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) = Q_{\nu\mu}(-\mathbf{k}, \omega, -\mathbf{B}_0) \tag{170}$$

Since (170) is now ω-independent, the external conductivity must satisfy the same symmetry relation

$$\hat{\sigma}_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) = \hat{\sigma}_{\nu\mu}(-\mathbf{k}, \omega, -\mathbf{B}_0) \tag{171}$$

in virtue of the FDT (114). In turn, Eq. (171), together with (20) and (15), gives similar relations for $\sigma_{\mu\nu}$ and $\epsilon_{\mu\nu}$. Moreover, since the system is invariant under spatial inversion, it follows that[30]

$$\hat{\sigma}_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) = \hat{\sigma}_{\nu\mu}(\mathbf{k}, \omega, -\mathbf{B}_0) \tag{172}$$

$$\sigma_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) = \sigma_{\nu\mu}(\mathbf{k}, \omega, -\mathbf{B}_0) \tag{173}$$

$$\epsilon_{\mu\nu}(\mathbf{k}, \omega, \mathbf{B}_0) = \epsilon_{\nu\mu}(\mathbf{k}, \omega, -\mathbf{B}_0) \tag{174}$$

On the basis of (174), we now establish a symmetry relation for the tensor components of $\epsilon_{\mu\nu}$. To do this, we observe that the space spanned by three coordinate axes in the directions of the *real* unit vectors $\mathbf{k}$, $\mathbf{k} \times \hat{\mathbf{B}}_0$, $\hat{\mathbf{B}}_0 \times \mathbf{k} \times \hat{\mathbf{B}}_0$, is a suitable representation

[30] See example, Ref. 1b, p. 483, Eq. (9.25); Ref. 2c, p. 581, Eq. (6.9); Ref. 13a, p. 143, Eq. (2.31).

for ε. Clearly, each member of such a dyadic tensor index refers to a direction along one of the axes in this space. Then a reversal in $\mathbf{B}_0$ leads to a reversal in the signature of the (dielectric) tensor component if one, and only one, index member points along $\mathbf{k} \times \hat{\mathbf{B}}_0$; otherwise, $\epsilon_{\mu\nu}$ is invariant under $\mathbf{B}_0$-reversal. This, together with the Onsager equation (174), results in the symmetry relation

$$\varepsilon(\mathbf{k}, \omega, \mathbf{B}_0) = \begin{Bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ -\epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & -\epsilon_{23} & \epsilon_{33} \end{Bmatrix} \tag{175}$$

in either the $\mathbf{k}$- or $\mathbf{B}_0$-system. Evidently, $\mathsf{Q}(\mathbf{k}, \omega, \mathbf{B}_0)$ satisfies a similar symmetry relation. In particular, if there is no dissipation, the dielectric tensor is Hermitian,[1b,8] so that (175) now reads

$$\varepsilon(\mathbf{k}, \omega, \mathbf{B}_0) = \begin{Bmatrix} \epsilon'_{11} & i\epsilon''_{12} & \epsilon'_{13} \\ -i\epsilon''_{12} & \epsilon'_{22} & i\epsilon''_{23} \\ \epsilon'_{13} & -i\epsilon''_{23} & \epsilon'_{33} \end{Bmatrix} \tag{176}$$

Finally, we note that the cold-plasma form

$$\varepsilon(\omega, \mathbf{B}_0) = \begin{Bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ -\epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{Bmatrix} \tag{177}$$

in the $\mathbf{B}_0$-system is a consequence of (175) and of the $\mathbf{k}$-independence.

In the case of the ion–electron plasma, $\mathsf{Q}^{(ee)}$ obeys the symmetry relation (170), while $\mathsf{Q}^{(ei)}$ does not (since $\mathsf{Q}^{(ei)}$ is not Hermitian). Therefore, while the sum of the electron and ion conductivities obeys the Onsager relation (172), one cannot assert from the FDT Eqs. (148) that $\hat{\sigma}^{(e)}$ and $\hat{\sigma}^{(i)}$ separately satisfy such symmetry rules (although this is certainly true in the case of models discussed in Ref. 21; a breakdown of symmetry cannot be expected in any approximation not of higher order than the Vlasov scheme or for $\mathbf{k} = 0$).

3.5. Kubo Sum-Rule Theorem

Starting from the FDT relation (114) for an electron plasma, one can generate an infinite set of relations between frequency moments of the dissipative part of the external conductivity and appropriate time derivatives of the equal-time current correlation tensor. This is the Kubo theorem,[13b] and it includes Eqs. (49), (52), and (56) for $\hat{\sigma}$ as lowest-order sum rules.

For the derivation of the Kubo theorem, it will be necessary to exploit the parity relations

$$Q_{(\mu\nu)}(\mathbf{k}, t, \mathbf{B}_0) = Q_{(\mu\nu)}(\mathbf{k}, t, -\mathbf{B}_0) = Q_{(\mu\nu)}(\mathbf{k}, -t, \mathbf{B}_0) \tag{178}$$

and

$$Q_{[\mu\nu]}(\mathbf{k}, t, \mathbf{B}_0) = -Q_{[\mu\nu]}(\mathbf{k}, t, -\mathbf{B}_0) = -Q_{[\mu\nu]}(\mathbf{k}, -t, \mathbf{B}_0) \tag{179}$$

A typical Kubo sum rule for, say, the symmetric part of $\hat{\sigma}$ can be generated by taking the frequency moment ω^{2n} (where $n = 0, 1, 2,...$) of the FDT (114). To illustrate, let $n = 1$, so that

$$\int_{-\infty}^{\infty} d\omega\, \omega^2 \hat{\sigma}'_{(\mu\nu)}(\mathbf{k}, \omega, \mathbf{B}_0) = \beta \int_{-\infty}^{\infty} d\omega\, \omega^2 \int_{-\infty}^{\infty} d\omega'\, \delta_+(\omega - \omega')\, Q_{(\nu\mu)}(\mathbf{k}, \omega', \mathbf{B}_0)$$
$$= -2\pi\beta \int_0^{\infty} dt\, Q_{(\nu\mu)}(\mathbf{k}, t, \mathbf{B}_0)\, \delta''(t)$$
$$= -\pi\beta \int_{-\infty}^{\infty} dt Q_{(\nu\mu)}(\mathbf{k}, t, \mathbf{B}_0)\, \delta''(t)$$
$$= -\pi\beta \left[\frac{\partial^2}{\partial t^2} Q_{(\mu\nu)}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} \tag{180}$$

Remark that we have used the parity rule (178) to arrive at this result.

Using the technique of (180), we now generate the hierarchy of f-sum rule moments for the four objects

$$\mathsf{\Delta}(\mathbf{k}, \omega) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathsf{\Delta}(\mathbf{k}, \omega), \qquad \mathbf{k} \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{k}, \qquad \frac{\alpha^{LL}(\mathbf{k}, \omega)}{\epsilon^{LL}(\mathbf{k}, \omega)}, \qquad \frac{\alpha^{TT}(\mathbf{k}, \omega)}{D^{TT}(\mathbf{k}, \omega)}$$

For the first of these, it is convenient to start from the FDT (120) written in the form

$$-2i\epsilon_0\omega\, \Delta_{\mu\alpha}(\mathbf{k}, \omega)\, \hat{D}_{\alpha\beta}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)\, \Delta_{\beta\nu}(\mathbf{k}, \omega) = \beta \int_{-\infty}^{\infty} dt\, e^{i\omega t} Q_{\nu\mu}(\mathbf{k}, t, \mathbf{B}_0) \tag{181}$$

Then, to obtain the hierarchy, multiply (181) by ω^{2n}, integrate the result over the entire frequency spectrum, and note, from the even parity in ω of the left-hand side and the parity relations (178) and (179) that only the symmetric tensor elements are relevant. For all positive integral values of n, including zero, the result is

$$\int_0^{\infty} d\omega\, \omega^{2n+1}\, \Delta_{\mu\alpha}(\mathbf{k}, \omega)\, \mathrm{Im}\, D_{(\alpha\beta)}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)\, \Delta_{\beta\nu}(\mathbf{k}, \omega) = (-)^n \frac{\pi\beta}{2\epsilon_0} \left[\frac{\partial^{2n}}{\partial t^{2n}} Q_{(\mu\nu)}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} \tag{182}$$

For the antisymmetric elements, one can similarly show that, for $n = 0, 1, 2,...,$

$$\int_0^{\infty} d\omega\, \omega^{2(n+1)}\, \Delta_{\mu\alpha}(\mathbf{k}, \omega)\, \mathrm{Re}\, D_{[\alpha\beta]}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)\, \Delta_{\beta\nu}(\mathbf{k}, \omega)$$
$$= (-)^n \frac{\pi\beta}{2\epsilon_0} \left[\frac{\partial^{2n+1}}{\partial t^{2n+1}} Q_{[\mu\nu]}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} \tag{183}$$

and, for $n = -1$, one obtains (63) in virtue of the FDT (114). Equations (182) and (183) are the generalized f-sum rules. Upon setting $n = 0$ and taking account of (94) and the evaluation of the equal-time current-correlation time derivative in Appendix C, we recover the nonrelativistic sum rules (62) and (64). For the corresponding relativstic rules, the right-hand sides of Eqs. (62) and (64) should be multiplied by the factors $\langle\gamma^{-1}[1 - (v^2/3c^2)]\rangle^0$ and $\langle\gamma^{-2}[1 - (2v^2/3c^2)]\rangle^0$, respectively [cf. Eq. (162) and Appendix C). One also recovers the $\hat{\sigma}$-sum rules reported by Kubo for quantum-mechanical systems[31] by inserting (60) and (61) into the above $n = 0$ case of (182) and (183). Apparently, the higher-order equal-time current correlation derivatives ($n \geqslant 1$) cannot be calculated without a knowledge of interactions. In connection with this, we recall from Section 2.3 that the corresponding higher-order sum rules do not exist, due to the eventual appearance of thermal terms of the order $\omega_0{}^2 k^2 v^2/\omega^4$ in the expansion similar to (67).

Let us briefly address ourselves to the evaluation of the relativistic brackets $\langle\gamma^{-1}[1-(v^2/3c^2)]\rangle^0$ and $\langle\gamma^{-2}[1-(2v^2/3c^2)]\rangle^0$. Since $\gamma = [1-(v^2/c^2)]^{-1/2}$ is the relative energy (rest plus kinetic) of an electron in the electromagnetic field, the ensemble-averaging operator takes the form

$$\langle\cdots\rangle^0 = \frac{\beta mc^2}{K_2(\beta mc^2)} \int_1^\infty d\gamma\, \gamma(\gamma^2-1)^{1/2}\, [\exp(-\beta mc^2\gamma)](\cdots)$$

for the equilibrium system,[29] where $K_2(\beta mc^2)$ is the modified Bessel function of order two. Letting[30]

$$G_n(\beta mc^2) = \frac{\int_1^\infty d\gamma\, \gamma^n(\gamma^2-1)^{1/2}\exp(-\beta mc^2\gamma)}{\int_1^\infty d\gamma\, \gamma(\gamma^2-1)^{1/2}\exp(-\beta mc^2\gamma)}$$

$$= \frac{\beta mc^2 \int_1^\infty d\gamma\, \gamma^n(\gamma^2-1)^{1/2}\exp(-\beta mc^2\gamma)}{K_2(\beta mc^2)}, \tag{184}$$

one can write

$$\begin{aligned}\langle\gamma^{-1}[1-(v^2/3c^2)]\rangle^0 &\equiv \tfrac{2}{3}\langle\gamma^{-1}\rangle^0 + \tfrac{1}{3}\langle\gamma^{-3}\rangle^0 \\ &= \tfrac{2}{3}G_0(\beta mc^2) + \tfrac{1}{3}G_{-2}(\beta mc^2)\end{aligned} \tag{185}$$

[where $G_0 = K_1(\beta mc^2)/K_2(\beta mc^2)$], and

$$\begin{aligned}\langle\gamma^{-2}[1-(2v^2/3c^2)]\rangle^0 &\equiv \tfrac{1}{3}\langle\gamma^{-2}\rangle^0 + \tfrac{2}{3}\langle\gamma^{-4}\rangle^0 \\ &= \tfrac{1}{3}G_{-1}(\beta mc^2) + \tfrac{2}{3}G_{-3}(\beta mc^2)\end{aligned} \tag{186}$$

To generate the Kubo hierarchy for the longitudinal projection

$$\mathbf{k}\cdot \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)\cdot\mathbf{k}$$

we multiply the FDT (121) by ω^{2n+1} and integrate over all ω. The result is

$$\int_0^\infty d\omega\, \omega^{2n+1}\mathbf{k}\cdot[\operatorname{Im} \mathsf{D}^{-1}(\mathbf{k}, \omega, \mathbf{B}_0)]\cdot\mathbf{k} = (-)^{n+1}\frac{\pi\beta}{2\epsilon_0 k^2}\left[\frac{\partial^{2(n+1)}}{\partial t^{2(n+1)}} P(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} \tag{187}$$

Equation (187) is the generalized longitudinal f-sum rule, which, for $n = 0$, reduces, as the case may be, to either the nonrelativistic rule (65) or its relativistic counterpart [right-hand side of (65) multiplied by $\langle\gamma^{-1}[1-(v^2/3c^2)]\rangle^0$].

[31] See Ref. 2c, p. 584, Eqs. (8.10), (8.19); Ref. 13a, pp. 155, 156, Eqs. (2.85), (2.87).

Now, suppose that there is no external magnetic field acting. Then Eq. (182) splits into the pure longitudinal and pure transverse f-sum rule moments:

$$-\int_0^\infty d\omega\, \omega^{2n+1} \operatorname{Im} \frac{1}{\epsilon^{LL}(\mathbf{k},\omega)} = \int_0^\infty d\omega\, \omega^{2n+1} \operatorname{Im} \frac{\alpha^{LL}(\mathbf{k},\omega)}{\epsilon^{LL}(\mathbf{k},\omega)} = (-)^{n+1} \frac{\pi\beta}{2\epsilon_0 k^2} \left[\frac{\partial^{2(n+1)}}{\partial t^{2(n+1)}} P(\mathbf{k},t)\right]_{t=0} \tag{188}$$

$$\int_0^\infty d\omega\, \omega^{2n-1}(k^2c^2 - \omega^2) \operatorname{Im} \frac{\alpha^{TT}(\mathbf{k},\omega)}{D^{TT}(\mathbf{k},\omega)} = (-)^{n+2} \frac{\pi\beta}{2\epsilon_0} \left[\frac{\partial^{2n}}{\partial t^{2n}} Q^{TT}(\mathbf{k},t)\right]_{t=0} \tag{189}$$

Remark that one could alternatively derive (188) from (187). Equation (189) can be stated in the more preferable form

$$\int_0^\infty d\omega\, \omega^{2n+1} \operatorname{Im} \frac{\alpha^{TT}(\mathbf{k},\omega)}{D^{TT}(\mathbf{k},\omega)} = \frac{\pi\beta}{2\epsilon_0} \sum_{s=0}^{n} (-)^{s+1} (kc)^{2(n-s)} \left[\frac{\partial^{2s}}{\partial t^{2s}} Q^{TT}(\mathbf{k},t)\right]_{t=0}$$

by taking account of (70) and (71). Upon setting $n = 0$, Eqs. (188) and (189) reduce to the rules (66) and (71) with the right-hand sides multiplied by the factor $\langle \gamma^{-1}[1 - (v^2/3c^2)]\rangle^0$ for the relativistic case.

In Appendix D, we present a brief review of Kubo's sum-rule expansion technique[13b] for the external conductivity. The resulting infinite sets generated therein are identical to those generated by the FDT moment method. Concerning the denominator expansion of the Kramers–Kronig relations given in this appendix, it is especially important to remember that the extent of each such expansion *may* be profoundly limited by the particular choice of plasma model and by the ensuing eventual divergence of the frequency-moment integrals (see footnote 14). Consequently, higher-order frequency moments of the dissipative parts of the external conductivitiy tensor and their corresponding equal-time current correlation time derivatives may very well not exist in the Kubo sum-rule theorem.

APPENDIX A

In order to see in detail the boundedness of $\sigma(\mathbf{k}, \omega = 0, \mathbf{B}_0)$ and of $\hat{\sigma}(\mathbf{k}, \omega = 0, \mathbf{B}_0)$ and $\mathsf{D}^{-1}(\mathbf{k}, \omega = 0, \mathbf{B}_0)$, we cite separately the zero-frequency results of four simple plasma models.

Model 1. Warm, Collisionless Electron Plasma (Vlasov Plasma)[21]

The $\omega \to 0$ behavior in the $\mathbf{k}$-system $[\mathbf{k} = (0, 0, k), \mathbf{B}_0 = (B_{0x}, 0, B_{0z})]$ can be shown to be

$$\sigma(\mathbf{k}, \omega \to 0, \mathbf{B}_0) \to \begin{Bmatrix} \text{const} & \text{const} & \sim i\omega \\ & \text{const} & \sim i\omega \\ & & \sim -i\omega\, \kappa^2/k^2 \end{Bmatrix} \tag{A.1}$$

$$\mathsf{D}^{-1}(\mathbf{k}, \omega \to 0, \mathbf{B}_0) \to \begin{Bmatrix} \sim\omega^2 & \sim i\omega^3 & \sim\omega^2 \\ & \sim\omega^2 & \sim\omega^2 \\ & & -k^2/(k^2+\kappa^2) \end{Bmatrix}, \tag{A.2}$$

Clearly,

$$\alpha^{LL}(\mathbf{k}, \omega \to 0) \to \kappa^2/k^2 \tag{A.3}$$

and, in the zero-frequency limit, those components of the electric polarizability which correspond to the constant elements of σ must diverge as $1/\omega$. The longitudinal conductivity is, however, zero, and the longitudinal polarizability is bounded in this model.

Model 2a. Cold, Collisional Ion–Electron Plasma with Constant Collision Frequency ν

It is understood in this model that the collisions take place as if they were generated by fixed scattering centers (no drift). As ω tends to zero, one finds, in the $\mathbf{B}_0$-system [$\mathbf{k} = (k_x, 0, k_z)$, $\mathbf{B}_0 = (0, 0, \mathbf{B}_0)$], that

$$\sigma \to \sigma_0^{(e)} \begin{Bmatrix} \dfrac{\nu^2}{\nu^2+\omega_c{}^2} & \dfrac{\nu^3/\omega_c}{\nu^2+\omega_c{}^2} & 0 \\ & \dfrac{\nu^2}{\nu^2+\omega_c{}^2} & 0 \\ & & 1 \end{Bmatrix} + \sigma_0^{(i)} \begin{Bmatrix} \dfrac{\nu^2}{\nu^2+\Omega_c{}^2} & \dfrac{-\nu^3/\Omega_c}{\nu^2+\Omega_c{}^2} & 0 \\ & \dfrac{\nu^2}{\nu^2+\Omega_c{}^2} & 0 \\ & & 1 \end{Bmatrix}$$

$$+ \begin{Bmatrix} a_1\omega & -a_2\omega^2 & 0 \\ & a_1\omega & 0 \\ & & i\dfrac{\omega}{\nu}\sigma_0 \end{Bmatrix} \tag{A.4}$$

where

$$\sigma_0 = \sigma_0^{(e)} + \sigma_0^{(i)} = \frac{\epsilon_0}{\nu}(\omega_0{}^2 + \Omega_0{}^2)$$

$$a_1 = -i\epsilon_0(\omega_0{}^2 + \Omega_0{}^2)/\omega_c\Omega_c, \qquad a_2 = -\epsilon_0(\omega_0{}^2 + \Omega_0{}^2)(\omega_c - \Omega_c)/\omega_c{}^2\Omega_c{}^2 \tag{A.5}$$

Note that all the leading terms are constant, and, thus, the longitudinal polarizability will not be bounded any more.

Model 2b. Cold, Collisional Electron–Ion Plasma with Infinitely Heavy Ions

The dc σ for the electron plasma is given by

$$\sigma(\mathbf{k}, \omega = 0, \mathbf{B}_0) = \sigma_0^{(e)} \begin{Bmatrix} \nu^2/(\nu^2+\omega_c{}^2) & -\nu\omega_c/(\nu^2+\omega_c{}^2) & 0 \\ & \nu^2/(\nu^2+\omega_c{}^2) & 0 \\ & & 1 \end{Bmatrix} \tag{A.6}$$

Although there is a change in the off-diagonal elements, this does not alter the previous conclusion. It can be shown, for both cases, however, that the *external* conductivity vanishes:

$$\hat{\sigma}^{LL}(\mathbf{k}, \omega = 0, \mathbf{B}_0) = 0 = \hat{\sigma}^{TL}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \tag{A.7}$$

In fact, this is a general result valid for any classical plasma model (see footnote 12).

All the elements of D^{-1} are still at least of order ω:

$$\mathsf{D}^{-1}(\mathbf{k}, \omega \to 0, \mathbf{B}_0) \to \begin{Bmatrix} \sim i\omega & \sim\omega^2 & \sim i\omega \\ & \sim\omega^2 & \sim\omega^2 \\ & & \sim i\omega \end{Bmatrix} \tag{A.8}$$

Model 3. Cold, Collisional Ion–Electron Plasma with Constant Electron–Ion Collision Frequency[22]

This is a more realistic model than Model 2a: The collision process is described as taking place between electrons and ions following their respective trajectories (the main difference originating from the common drift of the electrons and ions). In the $\mathbf{B}_0$-system, one finds that, as $\omega \to 0$,

$$\sigma \to \begin{pmatrix} a_1\omega & -a_2\omega^2 & 0 \\ & a_1\omega & 0 \\ & & \sigma_0 \end{pmatrix} + i\frac{\omega}{\nu_{\text{eff}}} \begin{pmatrix} \frac{\nu_{\text{eff}}^2}{\omega_c\Omega_c} a_1\omega & -2\frac{\nu_{\text{eff}}^2}{\omega_c\Omega_c} a_2\omega^2 & 0 \\ & \frac{\nu_{\text{eff}}^2}{\omega_c\Omega_c} a_1\omega & 0 \\ & & \sigma_0 \end{pmatrix} \tag{A.9}$$

where σ_0, a_1, and a_2 are defined in (A.5) with ν replaced therein by an effective ion–electron collision frequency ν_{eff}. Due to the common drift, the dc conductivities vanish as in the collisionless model. Consequently, some elements of D^{-1} also disappear faster:

$$\mathsf{D}^{-1}(\mathbf{k}, \omega \to 0, \mathbf{B}_0) \to \begin{Bmatrix} \sim i\omega & \sim\omega^4 & \sim i\omega \\ & \sim\omega^2 & \sim\omega^4 \\ & & \sim i\omega \end{Bmatrix} \tag{A.10}$$

One can readily convince oneself that, in the above models, all the elements of $\mathfrak{a}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \cdot \mathsf{D}^{-1}(\mathbf{k}, \omega = 0, \mathbf{B}_0)$ are bounded.

APPENDIX B

If the customary plasma expansion scheme (expansion in e^2n and in e^2 regarded as independent) is employed to calculate $\mathfrak{a}$, the sum rules with $n = 0, -1$ in Table II are obviously satisfied independently to any order, while the sum rules with $n = 1, 2$ are exhausted in the Vlasov approximation ($[e^2]^0$), since the constant values of the integrals are of the same order; the integrals of the higher-order contributions should yield zero.

We now present some detailed calculations to show that, for a warm, collisionless electron plasma in a constant external magnetic fiield, the (Vlasov) components of $\mathfrak{a}$[19,21] indeed behave in the expected way. Consider first the element (written in the $\mathbf{B}_0$-system)

$$\alpha_{11} = (\omega_0^2/\omega^2)\, \zeta_0(e^{-\mu}/\mu) \sum_{n=-\infty}^{\infty} n^2 I_n(\mu)\, Z(\zeta_n)$$

where

$$\zeta_n = \zeta_0 - n\zeta_c\,, \qquad \zeta_0 = \left(\frac{m\beta}{2}\right)^{1/2} \frac{\omega}{k_z}\,, \qquad \zeta_c = \left(\frac{m\beta}{2}\right)^{1/2} \frac{\omega_c}{k_z}\,, \qquad \mu = \frac{k_x^2}{m\beta\omega_c^2}$$

and

$$\begin{aligned} Z(\zeta_n) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta - \zeta_n} \exp(-\zeta_n^2) \\ &= \frac{1}{\sqrt{\pi}} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta - \zeta_n} \exp(-\zeta_n^2) + i\sqrt{\pi} \exp(-\zeta_n^2) \end{aligned}$$

is the plasma dispersion function, with $I_n(\mu)$ the modified Bessel function of order n. The calculation proceeds as follows:

$$\begin{aligned} \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha''_{11}(\mathbf{k}, \omega, \mathbf{B}_0) &= \frac{1}{\sqrt{\pi}} \frac{\omega_0^2 e^{-\mu}}{\mu} \sum n^2 I_n(\mu) \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^3} \zeta_0 \exp(-\zeta_n^2) \\ &= \frac{\omega_0^2 \beta m}{k_z^2} \frac{e^{-\mu}}{2\mu} \sum n^2 I_n(\mu) \frac{1}{\sqrt{\pi}} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\zeta_n}{(\zeta_n + n\zeta_c)^2} \exp(-\zeta_n^2) \\ &= \frac{\omega_0^2 \beta m}{k_z^2} \frac{e^{-\mu}}{2\mu} \sum n^2 I_n(\mu) \operatorname{Re} Z'(-n\zeta_c) \end{aligned} \tag{B.1}$$

On the other hand,

$$\alpha_{11}(\mathbf{k}, \omega \to 0, \mathbf{B}_0) \to \frac{\omega_0^2}{\omega^2} \zeta_0 \frac{e^{-\mu}}{\mu} \sum n^2 I_n(\mu)\, Z(-n\zeta_c) + \frac{\omega_0^2 \beta m}{k_z^2} \frac{e^{-\mu}}{2\mu} \sum n^2 I_n(\mu)\, Z'(-n\zeta_c)$$

and, since $\operatorname{Re} Z(-n\zeta_c)$ has odd parity in n, one finds that

$$\alpha'_{11}(\mathbf{k}, \omega \to 0, \mathbf{B}_0) \to \frac{\omega_0^2 \beta m}{k_z^2} \frac{e^{-\mu}}{2\mu} \sum n^2 I_n(\mu) \operatorname{Re} Z'(-n\zeta_c) \tag{B.2}$$

Comparison of (B.1) and (B.2) then yields the sum rule

$$\frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha''_{11}(\mathbf{k}, \omega, \mathbf{B}_0) = \alpha'_{11}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \tag{B.3}$$

For the remaining elements of $\boldsymbol{\alpha}$ (in the $\mathbf{B}_0$-system), one can similarly show that

$$\begin{aligned} \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha''_{22}(\mathbf{k}, \omega, \mathbf{B}_0) &= \frac{\omega_0^2 \beta m}{k_z^2} \frac{e^{-\mu}}{2\mu} \sum \operatorname{Re} Z'(-n\zeta_c) \left[\left(n^2 + 2\mu^2 - 2\mu^2 \frac{d}{d\mu}\right) I_n(\mu)\right] \\ &= \alpha'_{22}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \end{aligned} \tag{B.4}$$

$$\begin{aligned} \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \alpha''_{33}(\mathbf{k}, \omega, \mathbf{B}_0) &= \frac{\omega_0^2 \beta m}{k_z^2} \Big\{ 1 - 2e^{-\mu} \sum I_n(\mu)\, n\zeta_c \operatorname{Re} Z(-n\zeta_c) \\ &\qquad + e^{-\mu} \sum I_n(\mu)\, n^2 \zeta_c^2 \operatorname{Re} Z'(-n\zeta_c) \Big\} \\ &= \alpha'_{33}(\mathbf{k}, \omega = 0, \mathbf{B}_0) \end{aligned} \tag{B.5}$$

$$\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\,\alpha''_{13}(\mathbf{k},\omega,\mathbf{B}_0) = \frac{\omega_0{}^2\beta m}{k_z{}^2}\frac{e^{-\mu}}{(2\mu)^{1/2}}\Big\{\sum nI_n(\mu)\,\mathrm{Re}\,Z(-n\zeta_c)$$
$$- \zeta_c\sum n^2I_n(\mu)\,\mathrm{Re}\,Z'(-n\zeta_c)\Big\}$$
$$= \alpha'_{13}(\mathbf{k},\omega=0,\mathbf{B}_0) \tag{B.6}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega\,\alpha'_{12}(\mathbf{k},\omega,\mathbf{B}_0) = \frac{\omega_0{}^2(\beta m)^{1/2}}{k_z\sqrt{2}}\,e^{-\mu}\sum n[I_n'(\mu)-I_n(\mu)]\,\mathrm{Re}\,Z(-n\zeta_c)$$
$$= -\epsilon_0^{-1}\sigma'_{12}(\mathbf{k},\omega=0,\mathbf{B}_0) \tag{B.7}$$

Observe also that, in the **k**-system,

$$\frac{1}{\pi}\mathscr{P}\int_{\infty}^{\infty}\frac{d\omega}{\omega}\,\alpha''_{LL}(\mathbf{k},\omega,\mathbf{B}_0) = \frac{1}{k^2}\{k_x{}^2\alpha'_{11}(\mathbf{k},\omega=0,\mathbf{B}_0)+2k_xk_z\alpha'_{13}(\mathbf{k},\omega=0,\mathbf{B}_0)$$
$$+ k_z{}^2\alpha'_{33}(\mathbf{k},\omega=0,\mathbf{B}_0)$$
$$= \kappa^2/k^2 \tag{B.8}$$

We next focus our attention on the sum rule (55). The calculation for the Vlasov α_{12} (in the $\mathbf{B}_0$-system) proceeds as follows:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega\,\omega^2\alpha'_{12}(\mathbf{k},\omega,\mathbf{B}_0) = \frac{1}{\sqrt{\pi}}\,\omega_0{}^2e^{-\mu}\sum n[I_n'(\mu)-I_n(\mu)]\int_{-\infty}^{\infty}d\omega\,\zeta_0\exp(-\zeta_n{}^2)$$
$$= \omega_0{}^2\omega_ce^{-\mu}\left(\frac{d}{d\mu}-1\right)\sum n^2I_n(\mu)$$
$$= \omega_0{}^2\omega_ce^{-\mu}\left(\frac{d}{d\mu}-1\right)\left(\mu^2\frac{d^2}{d\mu^2}+\mu\frac{d}{d\mu}-\mu^2\right)e^{\mu} = \omega_0{}^2\omega_c \tag{B.9}$$

where, in the third step, Bessel's equation is used to express the inconvenient $n^2I_n(\mu)$ in terms of the convenient $I_n(\mu)$.

Similarly, one finds that the Vlasov expressions for the symmetric elements of $\mathfrak{a}$ exhaust the sum rule (48).

APPENDIX C

Here, we present the detailed calculations for the equal-time correlation derivative,

$$\left[\frac{\partial}{\partial t}\,Q_{[\mu\nu]}(\mathbf{k},t,\mathbf{B}_0)\right]_{t=0} = \frac{1}{L^3}\left\langle j^0_{-\mathbf{k}\mu}(0)\,\frac{d}{dt}\,j^0_{\mathbf{k}\nu}(0)\right\rangle^0 \tag{C.1}$$

Equation (86) permits us to write (C.1) in the form

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} Q_{[\mu\nu]}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} \\
&\quad = \frac{e^2}{L^3} \sum_n \langle \dot{v}_{n\nu}(0)\, v_{n\mu}(0)\rangle^0 \\
&\quad - \frac{ie^2}{L^3} k_\alpha \sum_n \langle v_{n\nu}(0)\, v_{n\mu}(0)\, v_{n\alpha}(0)\rangle^0 \\
&\quad + \frac{e^2}{L^3} \sum_{m,n;m\neq n} \langle \dot{v}_{m\nu}(0)\, v_{n\mu}(0) \exp i\mathbf{k}\cdot[\mathbf{x}_n(0) - \mathbf{x}_m(0)]\rangle^0 \\
&\quad - \frac{ie^2}{L^3} k_\alpha \sum_{m,n;m\neq n} \langle v_{m\nu}(0)\, v_{n\mu}(0)\, v_{m\alpha}(0) \exp i\mathbf{k}\cdot[\mathbf{x}_n(0) - \mathbf{x}_m(0)]\rangle^0 \qquad \text{(C.2)}
\end{aligned}$$

First, we note that all the odd velocity averages vanish, in view of the isotropy of the velocity distribution. The remaining second term also contributes nothing. To prove this, we first of all observe that

$$\begin{aligned}
&\frac{e^2}{L^3} \sum_{m,n;m\neq n} \langle \dot{v}_{m\nu}(0)\, v_{n\mu}(0) \exp i\mathbf{k}\cdot[\mathbf{x}_n(0) - \mathbf{x}_m(0)]\rangle^0 \\
&\quad = \frac{e^2}{L^3} \sum_{m,n;m\neq n} \int d\mathbf{\Gamma}^c\, \Omega^0 \dot{v}_{m\nu} [\exp i\mathbf{k}\cdot(\mathbf{x}_n - \mathbf{x}_m)] \frac{\partial H^0}{\partial p_{n\mu}} \\
&\quad = -\frac{e^2}{\beta L^3} \sum_{m,n;m\neq n} \int d\mathbf{\Gamma}^c\, \dot{v}_{m\nu} \frac{\partial}{\partial p_{n\mu}} [\Omega^0 \exp i\mathbf{k}\cdot(\mathbf{x}_n - \mathbf{x}_m)] \\
&\quad = \frac{e^2}{L^3} \sum_{m,n;m\neq n} \int d\mathbf{\Gamma}^c\, \Omega^0 [\exp i\mathbf{k}\cdot(\mathbf{x}_n - \mathbf{x}_m)] \frac{\partial \dot{v}_{m\nu}}{\partial p_{n\mu}} \qquad \text{(C.3)}
\end{aligned}$$

where $d\mathbf{\Gamma}^c = d\{\mathbf{x}_i\}\, d\{\mathbf{p}_i\}\, d\{\mathbf{A}_\mathbf{q}^T\}\, d\{\mathbf{E}_\mathbf{q}^T\}$ indicates that, for convenience, we choose, in this appendix, to ensemble-average in the Coulomb gauge. Now, from the relativistic relation (190),

$$\begin{aligned}
\dot{v}_{m\nu} = \frac{1}{m\gamma_m}\left(\delta_{\nu\lambda} - \frac{v_{m\nu}v_{m\lambda}}{c^2}\right)\Bigg\{ &\dot{p}_{m\lambda} + \frac{e}{L^3}\sum_\mathbf{q} (E^T_{-\mathbf{q}\lambda} + iq_\alpha v_{m\alpha} A^T_{-\mathbf{q}\lambda})(\exp -i\mathbf{q}\cdot\mathbf{x}_m) \\
&- ev_{m\alpha} \frac{\partial A_{0\lambda}(\mathbf{x}_m)}{\partial x_{m\alpha}}\Bigg\} \qquad \text{(C.4)}
\end{aligned}$$

for the mth particle. Hence,

$$\frac{\partial \dot{v}_{m\nu}}{\partial p_{n\mu}} = \frac{1}{m\gamma_m}\left(\delta_{\nu\lambda} - \frac{v_{m\nu}v_{m\lambda}}{c^2}\right)\frac{\partial \dot{p}_{m\lambda}}{\partial p_{n\mu}} = -\frac{1}{m\gamma_m}\left(\delta_{\nu\lambda} - \frac{v_{m\nu}v_{m\lambda}}{c^2}\right)\frac{\partial v_{n\mu}}{\partial x_{m\lambda}} = 0$$

since $\mathbf{v}_n = \mathbf{v}_n(\mathbf{p}_n, \mathbf{x}_n, \{\mathbf{A}_\mathbf{q}^T\})$. Equation (C.1) now reads

$$\left[\frac{\partial}{\partial t} Q_{[\mu\nu]}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} = \frac{e^2}{L^3}\sum_n \langle \dot{v}_{n\nu}(0)\, v_{n\mu}(0)\rangle^0 \qquad \text{(C.5a)}$$

$$= \frac{n_0 e^2}{\beta}\left\langle \frac{\partial \dot{v}_{n\nu}(0)}{\partial p_{n\mu}(0)}\right\rangle^0 \qquad \text{(C.5b)}$$

We now evaluate this expression both for a nonrelativistic and for a relativistic plasma.

Nonrelativistic Plasma

Starting from the relation

$$v_n = \frac{1}{m}\left[p_{n\nu} - \frac{e}{L^3}\sum_{\mathbf{q}} A^T_{-\mathbf{q}\nu}(t)(\exp -i\mathbf{q}\cdot\mathbf{x}_n) - eA_{0\nu}(\mathbf{x}_n)\right] \tag{C.6}$$

in the Coulomb gauge, one obtains

$$\dot{v}_{n\nu} = \frac{1}{m}\left[\dot{p}_{n\nu} + \frac{e}{L^3}\sum_{\mathbf{q}} (E^T_{-\mathbf{q}\nu} + iv_{n\lambda}q_\lambda A^T_{-\mathbf{q}\nu})(\exp -i\mathbf{q}\cdot\mathbf{x}_n) - ev_{n\lambda}\frac{\partial A_{0\nu}}{\partial x_{n\lambda}}\right] \tag{C.7}$$

whence

$$\begin{aligned}
\frac{\partial \dot{v}_{n\nu}}{\partial p_{n\mu}} &= \frac{1}{m}\left[\frac{\partial \dot{p}_{n\nu}}{\partial p_{n\mu}} + \frac{ie}{L^3}\frac{\partial v_{n\lambda}}{\partial p_{n\mu}}\sum_{\mathbf{q}} q_\lambda A^T_{-\mathbf{q}\nu}(\exp -i\mathbf{q}\cdot\mathbf{x}_n) - e\frac{\partial v_{n\lambda}}{\partial p_{n\mu}}\frac{\partial A_{0\nu}}{\partial x_{n\lambda}}\right] \\
&= \frac{1}{m}\left[-\frac{\partial v_{n\mu}}{\partial x_{n\nu}} - \frac{e}{mL^3}\frac{\partial}{\partial x_{n\mu}}\sum_{\mathbf{q}} A^T_{-\mathbf{q}\nu}(\exp -i\mathbf{q}\cdot\mathbf{x}_n) - \frac{e}{m}\frac{\partial A_{0\nu}}{\partial x_{n\mu}}\right] \\
&= \frac{e}{m^2}\left(\frac{\partial A_\mu{}^T(\mathbf{x}_n)}{\partial x_{n\nu}} - \frac{\partial A_\nu{}^T(\mathbf{x}_n)}{\partial x_{n\mu}}\right) + \frac{e}{m^2}\left(\frac{\partial A_{0\mu}(\mathbf{x}_n)}{\partial x_{m\nu}} - \frac{\partial A_{0\nu}(\mathbf{x}_n)}{\partial x_{n\mu}}\right) \\
&= -\frac{e}{m^2}\epsilon_{\mu\nu\alpha}B_{n\alpha} - \frac{e}{m^2}\epsilon_{\mu\nu\alpha}B_{0\alpha}
\end{aligned} \tag{C.8}$$

where $\mathbf{B}_n$ is the microscopic magnetic field. We note that $\langle B_n\rangle^0 = 0$ because (1) the classical plasma is not diamagnetic, and (2) the system is assumed to be large ($L^3 \to \infty$), so that there can be no surface currents on its remote boundaries. Hence, for the nonrelativistic classical plasma, the equal-time current-correlation derivative is

$$\left[\frac{\partial}{\partial t} Q_{[\mu\nu]}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} = \frac{\omega_0{}^2\omega_c}{\beta}\epsilon_0\epsilon_{[\mu\nu]\alpha}\hat{B}_{0\alpha} \tag{C.9}$$

where $\omega_c = |e|B_0/m$.

Relativistic Plasma

The relativistic calculation proceeds more easily if one starts from (C.5a) and the *equation of motion,*

$$\begin{aligned}
\dot{v}_{n\nu}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{p}_n, \{\mathbf{A}_\mathbf{q}{}^T\}, \{\mathbf{E}_\mathbf{q}{}^T\}) = \frac{e}{m\gamma_n}\left(\delta_{\nu\lambda} - \frac{v_{n\nu}v_{n\lambda}}{c^2}\right)\{E^T_{n\lambda}(\mathbf{x}_n, \{\mathbf{E}_\mathbf{q}{}^T\}) + E^L_{n\lambda}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \\
+ \epsilon_{\lambda\alpha\beta}v_{n\alpha}[B_{n\beta}(\mathbf{x}_n, \{\mathbf{A}_\mathbf{q}{}^T\}) + B_{0\beta}]\}
\end{aligned} \tag{C.10}$$

Then

$$\begin{aligned}
\langle \dot{v}_{n\nu}v_{n\mu}\rangle^0 &= \frac{e}{m}\left\langle \gamma_n^{-1}\left(\delta_{\nu\lambda} - \frac{v_{n\nu}v_{n\lambda}}{c^2}\right)v_{n\mu}E^T_{n\lambda}\right\rangle^0 \\
&+ \frac{e}{m}\left\langle \gamma_n^{-1}\left(\delta_{\nu\lambda} - \frac{v_{n\nu}v_{n\lambda}}{c^2}\right)v_{n\mu}E^L_{n\lambda}\right\rangle^0 + \frac{e}{m}\epsilon_{\nu\alpha\beta}\langle\gamma_n^{-1}v_{n\mu}v_{n\alpha}(B_{n\beta} + B_{0\beta})\rangle^0
\end{aligned} \tag{C.11}$$

The first and second terms on the right-hand side vanish on account of the independence of the equilibrium velocity-distribution function both of the transverse field coordinates and of the position coordinates which determine the longitudinal field, and in virtue of the vanishing of the average electric field. Only the correlation

$$\begin{aligned}
\frac{e}{m}\,\epsilon_{\nu\alpha\beta}\langle\gamma_n^{-1}v_{n\mu}v_{n\alpha}(B_{n\beta}+B_{0\beta})\rangle^0 &= -\frac{e}{\beta m}\,\epsilon_{\nu\alpha\beta}\int d\Gamma^c\,\gamma_n^{-1}v_{n\alpha}\,\frac{\partial}{\partial p_{n\mu}}\{\Omega^0(B_{n\beta}+B_{0\beta})\}\\
&= \frac{e}{\beta m}\,\epsilon_{\nu\alpha\beta}\int d\Gamma^c\,\Omega^0(B_{n\beta}+B_{0\beta})\,\frac{\partial}{\partial p_{n\mu}}(\gamma_n^{-1}v_{n\alpha})\\
&= \frac{e}{\beta m^2}\,\epsilon_{\nu\alpha\beta}\left\langle\gamma_n^{-2}\left(\delta_{\alpha\mu}-2\,\frac{v_{n\alpha}v_{n\mu}}{c^2}\right)(B_{n\beta}+B_{0\beta})\right\rangle^0\\
&= \frac{-\omega_c}{\beta m}\,\epsilon_{\nu\alpha\beta}\hat{B}_{0\beta}\left\langle\gamma_n^{-2}\left(\delta_{\alpha\mu}-2\,\frac{v_{n\alpha}v_{n\mu}}{c^2}\right)\right\rangle^0\\
&= \frac{-\omega_c}{\beta m}\,\epsilon_{[\nu\mu]\beta}\hat{B}_{0\beta}\left\langle\gamma_n^{-2}\left(1-\frac{2v^2}{3c^2}\right)\right\rangle^0
\end{aligned}$$

survives in (C.11). Note that we have again exploited the independence of the transverse field coordinates and the fact that $\langle B_{n\beta}\rangle^0 = 0$ for the large classical system. Then, ultimately, the equal-time current-correlation derivative is given by

$$\left[\frac{\partial}{\partial t}\,Q_{[\mu\nu]}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} = -\frac{\omega_0{}^2\omega_c}{\beta}\,\epsilon_0\epsilon_{[\mu\nu]\lambda}\hat{B}_{0\lambda}\left\langle\gamma^{-2}\left(1-\frac{2v^2}{3c^2}\right)\right\rangle^0 \tag{C.12}$$

for the relativistic plasma.

APPENDIX D—SUM-RULE EXPANSION

Repeated integrations of the FDT (114) by parts gives

$$\begin{aligned}
\hat{\sigma}_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0) = &-\frac{\beta}{(i\omega)}\,Q_{\mu\nu}(\mathbf{k}, t=0, \mathbf{B}_0) + \frac{\beta}{(i\omega)^2}\left[\frac{\partial}{\partial t}\,Q_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0}\\
&-\frac{\beta}{(i\omega)^3}\left[\frac{\partial^2}{\partial t^2}\,Q_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} + \cdots + \frac{(-)^{n+1}\beta}{(i\omega)^{n+1}}\left[\frac{\partial^n}{\partial t^n}\,Q_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0}\\
&+\frac{(-)^{n+1}\beta}{(i\omega)^{n+1}}\int_0^\infty dt\,e^{i\omega t}\,\frac{\partial^{n+1}}{\partial t^{n+1}}\,Q_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0)
\end{aligned} \tag{D.1}$$

Now, the reality of $Q_{\mu\nu}(\mathbf{r}, t, \mathbf{B}_0)$, together with invariance under spatial inversion, leads to the reality of $Q_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0)$. Then, upon equating real and imaginary parts in (D.1), one obtains

$$\hat{\sigma}'_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0) = -\frac{\beta}{\omega^2}\left[\frac{\partial}{\partial t}\,Q_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} + \frac{\beta}{\omega^4}\left[\frac{\partial^3}{\partial t^3}\,Q_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} - \cdots \tag{D.2}$$

$$\hat{\sigma}''_{\nu\mu}(\mathbf{k}, \omega, \mathbf{B}_0) = \frac{\beta}{\omega}\,Q_{\mu\nu}(\mathbf{k}, t=0, \mathbf{B}_0) - \frac{\beta}{\omega^3}\left[\frac{\partial^2}{\partial t^2}\,Q_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} + \cdots \tag{D.3}$$

In the high-frequency limit ($\omega \to \infty$), the usual denominator expansion of the Kramers–Kronig relations yields (see footnote 13)

$$\hat{\sigma}'_{[\mu\nu]}(\mathbf{k}, \omega \to \infty, \mathbf{B}_0) = -\frac{1}{\pi\omega^2}\int_{-\infty}^{\infty} d\omega'\, \omega' \hat{\sigma}''_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) - \frac{1}{\pi\omega^4}\int_{-\infty}^{\infty} d\omega'\, \omega'^3 \hat{\sigma}''_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) - \cdots \tag{D.4}$$

$$\hat{\sigma}''_{(\mu\nu)}(\mathbf{k}, \omega \to \infty, \mathbf{B}_0) = \frac{1}{\pi\omega}\int_{-\infty}^{\infty} d\omega' \hat{\sigma}'_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) + \frac{1}{\pi\omega^3}\int_{-\infty}^{\infty} d\omega'\, \omega'^2 \hat{\sigma}'_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) + \cdots \tag{D.5}$$

Then, in an obvious comparison of (D.2) and (D.3) with (D.4) and (D.5), we see that

$$\int_0^{\infty} d\omega'\, \omega' \hat{\sigma}_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) = \frac{-\pi\beta}{2}\left[\frac{\partial}{\partial t} Q_{[\mu\nu]}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} \tag{D.6a}$$

$$\int_0^{\infty} d\omega'\, \omega'^3 \hat{\sigma}''_{[\mu\nu]}(\mathbf{k}, \omega', \mathbf{B}_0) = \frac{\pi\beta}{2}\left[\frac{\partial^3}{\partial t^3} Q_{[\mu\nu]}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} \tag{D.6b}$$

and

$$\int_0^{\infty} d\omega'\, \hat{\sigma}'_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) = \frac{\pi\beta}{2} Q_{(\mu\nu)}(\mathbf{k}, t = 0, \mathbf{B}_0) \tag{D.7a}$$

$$\int_0^{\infty} d\omega'\, \omega'^2 \hat{\sigma}'_{(\mu\nu)}(\mathbf{k}, \omega', \mathbf{B}_0) = -\frac{\pi\beta}{2}\left[\frac{\partial^2}{\partial t^2} Q_{(\mu\nu)}(\mathbf{k}, t, \mathbf{B}_0)\right]_{t=0} \tag{D.7b}$$

NOMENCLATURE

$\mathbf{E}(\mathbf{k}, \omega)$	electric field intensity
$\hat{\mathbf{E}}(\mathbf{k}, \omega)$	electric field in absence of plasma particles,
$\check{\mathbf{E}}(\mathbf{k}, \omega)$	electric field due to the plasma particles ($= \mathbf{E} - \hat{\mathbf{E}}$)
$\mathbf{B}(\mathbf{k}, \omega)$	magnetic induction
$\mathbf{D}(\mathbf{k}, \omega)$	electric induction
$\mathbf{H}(\mathbf{k}, \omega)$	magnetic field strength
$\mathbf{B}_0$	constant external magnetic field
$\mathbf{A}_0$	vector potential corresponding to $\mathbf{B}_0$
$\rho(\mathbf{k}, \omega)$, $\mathbf{j}(\mathbf{k}, \omega)$	charge and current densities due to the plasma particles
$\hat{\rho}(\mathbf{k}, \omega)$, $\hat{\mathbf{j}}(\mathbf{k}, \omega)$	charge and current densities of the external agency
$\boldsymbol{\varepsilon}(\mathbf{k}, \omega, \mathbf{B}_0)$	dielectric tensor of the plasma medium in the presence of $\mathbf{B}_0$
$\boldsymbol{\nu}(\mathbf{k}, \omega, \mathbf{B}_0)$	diamagnetic tensor
$\boldsymbol{\alpha}(\mathbf{k}, \omega, \mathbf{B}_0) =$	$\boldsymbol{\varepsilon}(\mathbf{k}, \omega, \mathbf{B}_0) - \mathbf{1}$, electric polarizability tensor
$\boldsymbol{\xi}(\mathbf{k}, \omega, \mathbf{B}_0)$	magnetic polarizability tensor
$\boldsymbol{\sigma}(\mathbf{k}, \omega, \mathbf{B}_0)$	ordinary conductivity tensor
$\hat{\boldsymbol{\sigma}}(\mathbf{k}, \omega, \mathbf{B}_0)$	external conductivity tensor

$\mathsf{D}(\mathbf{k}, \omega, \mathbf{B}_0) =$	$n^2\mathsf{T} - \varepsilon(\mathbf{k}, \omega, \mathbf{B}_0)$, dispersion tensor, where $\mathsf{T} = \mathsf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}$ is the transverse projection tensor ($\hat{\mathbf{k}}$ being the unit vector in the direction of $\mathbf{k}$) and $n = kc/\omega$ the index of refraction
$\Delta =$	$n^2\mathsf{T} - \mathsf{1}$, = vacuum wave operator (value of D in vacuum)
$\sigma^{\vee} =$	$\frac{1}{2}(\sigma + \sigma^{\dagger})$, Hermitian part of σ
$\sigma^{\wedge} =$	$\frac{1}{2}(\sigma - \sigma^{\dagger})$, Anti-Hermitian part of σ
σ', σ''	real and imaginary parts of σ
$R(\mathbf{r}, t)$	dissipated power per unit volume of plasma
U	total energy absorbed by the plasma
$R(\mathbf{k}, \omega) =$	$\mathbf{E}^*(\mathbf{k}, \omega) \cdot \sigma(\mathbf{k}, \omega, \mathbf{B}_0) \cdot \mathbf{E}(\mathbf{k}, \omega)$ corresponding spectral energy density
$W(\mathbf{r}, t) =$	$\frac{1}{2}\epsilon_0 E^2(\mathbf{r}, t) + (1/2\mu_0) B^2(\mathbf{r}, t)$, field energy density
$W(\mathbf{k}, \omega)$	$\frac{1}{2}\epsilon_0 \mathbf{E}^*(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) + (1/2\mu_0)\mathbf{B}^*(\mathbf{k}, \omega) \cdot \mathbf{B}(\mathbf{k}, \omega)$, energy content in a certain domain of $(\mathbf{k}, \omega)$-space for a single mode
$\mathbf{x}_i, \mathbf{p}_i, \mathbf{v}_i$	coordinate, momentum, and velocity of ith electron
$\gamma_i =$	$[1 - (v_i^2/c^2)]^{-1/2}$
$\mathbf{X}_j, \mathbf{P}_j, \mathbf{V}_j$	coordinate, momentum, and velocity of jth ion
$\{\mathbf{A_q}\}, \{\mathbf{E_q}\}$	field coordinates and momenta
$\rho_{\mathbf{k}}^0(t) =$	$e \sum_i \exp -i\mathbf{k} \cdot \mathbf{x}_i(t)$ — kth Fourier components of the microscopic electron charge and current densities of the system in equilibrium. Each electron carries charge $e = -\lvert e \rvert$
$\mathbf{j}_{\mathbf{k}}^0(t) =$	$e \sum_i \mathbf{v}_i \exp -i\mathbf{k} \cdot \mathbf{x}_i(t)$
$R_{\mathbf{k}}^0(t) =$	$-Ze \sum_j \exp -i\mathbf{k} \cdot \mathbf{X}_j(t)$ — kth Fourier components of the microscopic ion charge and current densities of the system in equilibrium. Each ion carries charge $Z\lvert e \rvert$
$\mathbf{J}_{\mathbf{k}}^0(t) =$	$-Ze \sum_j \mathbf{V}_j \exp i\mathbf{k} \cdot \mathbf{X}_j(t)$
$\mathbf{j}_{\mathbf{k}}'(t), \mathbf{J}_{\mathbf{k}}'(t)$	perturbations in the microscopic electron and ion current densities due to the presence of the small external vector potential agency $\hat{\mathbf{A}}(\mathbf{r}, t) = (1/L^3)\, \hat{\mathbf{A}}_{\mathbf{k}}(t) \exp i\mathbf{k} \cdot \mathbf{r}$
Ω	Liouville distribution function $= \Omega^0 + \Omega'$
Ω^0	macrocanonical distribution function characterizing the equilibrium state of the system in the infinite past
Ω'	small perturbation due to $\hat{\mathbf{A}}$
H^0	Hamiltonian of equilibrium system which includes interaction
H'	Hamiltonian for the interaction between the system and the small external perturbing agency $\hat{\mathbf{A}}$
$\langle \cdots \rangle^0 =$	$\int d\Gamma^R (\cdots)\, \Omega^0$, expectation value of any quantity over the equilibrium ensemble ($d\Gamma^R$ is an element of hypervolume in Γ-phase space)
$Z =$	$\int d\Gamma^R \exp(-\beta H^0)$
$P^{(ei)}(\mathbf{k}, t, \mathbf{B}_0) =$	$(1/L^3)\langle \rho_{\mathbf{k}}^0(t)\, R^0_{-\mathbf{k}}(0)\rangle^0$ — Electron–ion charge-density correlation function
$P^{(ie)}(\mathbf{k}, t, \mathbf{B}_0) =$	$(1/L^3)\langle R_{\mathbf{k}}^0(t)\, \rho^0_{-\mathbf{k}}(0)\rangle^0$
$Q^{(ei)}_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0) =$	$(1/L^3)\langle j^0_{\mathbf{k}\nu}(t)\, J^0_{-\mathbf{k}\mu}(0)\rangle^0$ — Electron–ion current-density correlation tensors
$Q^{(ie)}_{\mu\nu}(\mathbf{k}, t, \mathbf{B}_0) =$	$(1/L^3)\langle J^0_{\mathbf{k}\nu}(t)\, j^0_{-\mathbf{k}\mu}(0)\rangle^0$
$G(12)$	two-particle distribution function

$F(1)$	one-particle distribution function
$g(\mid x_2 - x_1 \mid) =$	$[G(12)/F(1)F(2)] - 1$, pair correlation function
N	total number of electron in volume L^3
n_0	equilibrium density (of electrons)
β^{-1}	temperature (in energy units)
$\omega_0 =$	$(n_0 e^2/m\epsilon_0)^{1/2}$, equilibrium electron plasma frequency
$\omega_c =$	$\mid e \mid B_0/m$, electron frequency
$\kappa^{-1} =$	$(\epsilon_0/\beta n_0 e^2)^{1/2}$, Debye length
$\Omega_0 =$	$(n_0 Z e^2/M\epsilon_0)^{1/2}$, equilibrium ion plasma frequency
$\Omega_c =$	ZeB_0/M, ion cyclotron frequency

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